# Average Range of Lipschitz Functions on Trees*

Yaokun Wu [†] Zeying Xu [‡] Yinfeng Zhu [§]

**Abstract**

In the study of graph indexed random walks, two conjectures on the average range of some functions on graphs and bipartite graphs are posed by Loebl-Nešetřil-Reed and by Benjamini-Häggström-Mossel, respectively. We verify these two conjectures for trees and present observations relevant to the original conjectures.

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## 1 Introduction

Let $V$ be a set. The *swapping map* $\mathrm{s}_V$ on $V \times V$ is the one which sends $(u, v) \in V \times V$ to $(v, u)$. It is natural to think of $(u, v) \in V \times V$ as the element $v - u$ in the linear space spanned by $V$ and so the swapping map is essentially the multiplication with $-1$.

A graph $G$ consists of a *vertex set* $\mathrm{V}(G)$, a *side set* $\mathrm{S}(G)$ together with a fixed-point free map $\mathrm{s}_G$ on $\mathrm{S}(G)$ such that $\mathrm{s}_G^2$ is the identity map, and a

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†Department of Mathematics and MOE-LSC, Shanghai Jiao Tong University, Shanghai 200240, China. E-mail: ykwu@sjtu.edu.cn.

‡Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China. E-mail: zane_xu@sjtu.edu.cn.

§Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China. E-mail: fengzi@sjtu.edu.cn.

*boundary map* $\partial_G$ which is an injective map from $\mathrm{S}(G)$ to $(\mathrm{V}(G) \times \mathrm{V}(G)) \setminus \{(v,v): v \in \mathrm{V}(G)\}$ satisfying

$$\partial_G \, \mathrm{s}_G = \mathrm{s}_{\mathrm{V}(G)} \, \partial_G.$$

If $\partial_G(s) = (a,b) = b - a$, we often simply denote the side $s$ by $\overrightarrow{ab}$. The *edge set* $\mathrm{E}(G)$ of $G$ is the set of orbits of $\mathrm{s}_G$. Indeed, if $e \in \mathrm{E}(G)$ corresponds to a pair of sides $\overrightarrow{ab}$ and $\overrightarrow{ba}$, we view $a$ and $b$ as the endpoints of the edge and denote it by $\overline{\partial}_G(e) = \{a,b\} \in \binom{\mathrm{V}(G)}{2}$. When only one graph is concerned, it is convenient to represent an edge $e$ with $\overline{\partial}_G(e) = \{a,b\}$ by $ab$. You may think of the two sides corresponding to one edge as the two directions running between its two endpoints. Unless stated otherwise, $\mathrm{V}(G)$, and hence $\mathrm{S}(G)$ and $\mathrm{E}(G)$, is always assumed to be a finite set. An *endomorphism* of $G$ is a map $\sigma$ from $\mathrm{V}(G)$ to itself such that $\sigma(a)\sigma(b) \in \mathrm{E}(G)$ for all $ab \in \mathrm{E}(G)$. We write $\mathrm{End}(G)$ for the set of all endomorphisms of $G$. A bijective map on $\mathrm{V}(G)$ is an *automorphism* of the graph $G$ if it falls into $\mathrm{End}(G)$.

The *order* of a graph $G$ is the size of its vertex set $\mathrm{V}(G)$. Let $\mathcal{G}_n$ denote the set of all order-$n$ connected graphs and let $\mathcal{T}_n$ denote the set of all order-$n$ trees. We write $K_n, S_n, P_n$ for the complete graph, the star tree and the path, respectively, of order $n$.

Let $\mathbb{R}$ be the set of real numbers. The set of *real* 0-*chains* on a graph $G$, denoted by $\mathrm{C}^0(G;\mathbb{R})$, is the real linear space consisting of all real functions on $\mathrm{V}(G)$. The set of *real* 1-*chains* on a graph $G$, denoted by $\mathrm{C}^1(G;\mathbb{R})$, is the set of those assignments $f$ of real values to the sides of $G$ such that

$$f(\overrightarrow{ab}) = f(b-a) = -f(a-b) = -f(\overrightarrow{ba})$$

for every $ab \in \mathrm{E}(G)$. The *coboundary operator* on $G$, denoted by $\mathrm{d}_G$, is the linear operator from $\mathrm{C}^0(G;\mathbb{R})$ to $\mathrm{C}^1(G;\mathbb{R})$ such that

$$\mathrm{d}_G(F)(\overrightarrow{ab}) := F(\partial_G(\overrightarrow{ab})) = F(b) - F(a)$$

for every $F \in \mathrm{C}^0(G;\mathbb{R})$ and $ab \in \mathrm{E}(G)$. The image of the coboundary operator $\mathrm{d}_G$ is the *cut space* of $G$. The elements of the cut space are also often called *potential difference functions* on $\mathrm{E}(G)$. For any two vertices $a$ and $b$ in the same connected component of $G$ and any $f \in \mathrm{C}^1(G;\mathbb{R})$, let $\int_{a\to b}^{G} f = f(\overrightarrow{ax_1}) + f(\overrightarrow{x_1x_2}) + \cdots + f(\overrightarrow{x_{t-1}x_t}) + f(\overrightarrow{x_tb})$, where $x_1, \ldots, x_t$ are chosen such that $ax_1, x_1x_2, \ldots, x_{t-1}x_t, x_tb$ are edges of $G$. It is easy to see that the value of $\int_{a\to b}^{G} f$ is independent with the choice of the path connecting $a$ and $b$, as

$$\int_{a\to b}^{G} f = F(b) - F(a)$$

provided $f = \mathrm{d}_G(F)$.

For every connected graph $G$ and every $f \in \mathrm{C}^1(G;\mathbb{R})$, we define the *range* of $f$ to be

$$\mathrm{r}_G(f) := \max_{a,b\in \mathrm{V}(G)} \int_{a\to b}^{G} f.$$

Choose a map $I$ from $\binom{\mathrm{V}(G)}{2}$ to $2^{\mathbb{R}}$. We write $\mathrm{C}^1_I(G;\mathbb{R})$ for the set of those potential differences $f$ on $G$ such that

$$\int_{a\to b}^{G} f \in I(ab)$$

for all $\{a,b\} \in \binom{\mathrm{V}(G)}{2}$. If $\mathrm{C}^1_I(G;\mathbb{R})$ is a finite set, let us define the *average range* of $\mathrm{C}^1_I(G;\mathbb{R})$ to be

$$\frac{\sum_{f\in \mathrm{C}^1_I(G;\mathbb{R})} \mathrm{r}_G(f)}{|\,\mathrm{C}^1_I(G;\mathbb{R})|},$$

which is a parameter of some interest in the study of the graph-indexed Markov chains. Note that the value of $I(ab)$ for $ab \in \mathrm{E}(G)$ may correspond to some continuous assumption, say when modelling some random surfaces in statistical physics, while those $I(ab)$ for $ab \notin \mathrm{E}(G)$ may be thought of as some long-range constraint or boundary condition [1, 2, 7, 11]. For example, grounded Lipschitz functions on a tree [10] are associated with the constraint that $I(ab) = \{0\}$ for all pairs of leaves $\{a,b\}$ of the tree.

The bulk of this article is concerned with $\mathrm{C}^1_{\mathcal{I}_G}(G;\mathbb{R})$, where $\mathcal{I}_G$ is given by

$$\mathcal{I}_G(ab) = \begin{cases} \{-1,0,1\}, & \text{if } ab \in \mathrm{E}(G), \\ \mathbb{R}, & \text{else.} \end{cases} \tag{1}$$

We call the elements of $\mathrm{C}^1_{\mathcal{I}_G}(G;\mathbb{R})$ *Lipschitz cuts* on $G$ and call the average range for $\mathrm{C}^1_{\mathcal{I}_G}(G;\mathbb{R})$ the *height* of $G$. We use $\mathscr{L}(G)$ for the set of Lipschitz cuts on $G$ and write $\mathrm{h}(G)$ for the height of $G$. For every $G \in \mathcal{G}_n$, the size of $\mathscr{L}(G)$ is at most $3^{n-1}$ and this bound is attained if and only if the graph $G$ is a tree. For any $a,b \in \mathrm{V}(G)$ and any nonnegative integer $r$, we use $\mathrm{h}^r_{a,b}(G)$ to denote the average range of those Lipschitz cuts $f \in \mathrm{C}^1(G;\mathbb{R})$ such that $\int_{a\to b}^{G} f \in \{r,-r\}$.

A *Lipschitz function* on a connected graph $G$ is any element $F \in \mathrm{C}^0(G;\mathbb{R})$ such that $\mathrm{d}(F)$ is a Lipschitz cut. In other words, $F \in \mathrm{C}^0(G;\mathbb{R})$ is Lipschitz if and only if, for all $a, b \in \mathrm{V}(G)$, $F(a) - F(b)$ takes an integer value not bigger than the distance between $a$ and $b$ in $G$. The *range* of the Lipschitz function $F$ is

$$\max F - \min F,$$

which coincides with the range of $\mathrm{d}(F) \in \mathrm{C}^1(G;\mathbb{R})$. Loebl, Nešetřil and Reed [9] initiated the study of the average range of all Lipschitz functions on a connected graph. It is easy to see that $K_n$ is the unique one from $\mathcal{G}_n$ with minimum height. In the other direction, we have the following conjecture posed by Loebl, Nešetřil and Reed.

**Conjecture 1.1.** *[9, Conjecture 1] For every positive integer $n$ and every $G \in \mathcal{G}_n$, $\mathrm{h}(G) \leq \mathrm{h}(P_n)$ holds.*

Intuitively but not precisely, one may expect more self-intersections in a random walk on a graph with less cutpoints and cutedges and larger diameter and so Conjecture 1.1 looks quite reasonable. In the following table, we display the heights of some paths of small order. Seems that the exact formula for the height of an $n$-path is unknown.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $\mathrm{h}(P_n)$ | $\frac{2}{3}$ | $\frac{10}{9}$ | $\frac{40}{27}$ | $\frac{146}{81}$ | $\frac{508}{243}$ | $\frac{1716}{729}$ | $\frac{5682}{2187}$ | $\frac{18546}{6561}$ |
| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | |
| $\mathrm{h}(P_n)$ | $\frac{59884}{19683}$ | $\frac{191744}{59049}$ | $\frac{609838}{177147}$ | $\frac{1928956}{531441}$ | $\frac{6073598}{1594323}$ | $\frac{19049962}{4782969}$ | $\frac{59553720}{14348907}$ | |

Let $G$ be a connected graph. Let us define a map $\widehat{\mathcal{I}}_G$ on $\binom{\mathrm{V}(G)}{2}$ by putting

$$\widehat{\mathcal{I}}_G(ab) = \begin{cases} \{-1, 1\}, & \text{if } ab \in \mathrm{E}(G), \\ \mathbb{R}, & \text{else,} \end{cases}$$

and refer to $\mathrm{C}^1_{\widehat{\mathcal{I}}_G}(G;\mathbb{R})$ by $\widehat{\mathscr{L}}(G)$. The set of integers can be thought of as an infinite path where every two consecutive integers are adjacent. With this understanding, $\widehat{\mathscr{L}}(G)$ is nothing but the coboundaries of those homomorphisms from $G$ to the infinite path [1, 9]. Note that $\widehat{\mathscr{L}}(G)$ is nonempty if and only if $G$ is bipartite. For a connected bipartite graph $G$, let us denote by $\widehat{\mathrm{h}}(G)$ the average range of $\widehat{\mathscr{L}}(G)$, namely

$$\widehat{\mathrm{h}}(G) := \frac{\sum_{f\in\widehat{\mathscr{L}}(G)} \mathrm{r}_G(f)}{|\widehat{\mathscr{L}}(G)|}.$$

Regarding this parameter, Benjamini, Häggström and Mossel made the following conjecture, which is a main motivation for Loebl, Nešetřil and Reed to propose their Conjecture 1.1.

**Conjecture 1.2.** *[1, Conjecture 2.2] Let $n$ be a positive number and let $G$ be a connected bipartite graph of order $n$. Then, $\widehat{\mathrm{h}}(G) \leq \widehat{\mathrm{h}}(P_n)$.*

It is clear that almost all questions on $\mathrm{h}(G)$ have their counterparts for $\widehat{\mathrm{h}}(G)$. It will be interesting if for some natural questions these two parameters turn out to have different behaviours. We will prove Conjecture 1.1 and Conjecture 1.2 for trees in this paper; see Corollary 2.6 and Corollary 2.12. Note that our proofs for these two results are almost the same and so we do not bother to give full details for the latter one. Actually, in many places of this paper, we will only discuss the parameter $\mathrm{h}(G)$, leaving aside the corresponding discussions on $\widehat{\mathrm{h}}(G)$.

Csikvári and Lin showed that among all order-$n$ trees, the star tree $S_n$ has the largest number of endomorphisms and the path $P_n$ has the smallest number of endomorphisms [6, Theorem 1.8]. For any $f \in \mathrm{C}^1(G;\mathbb{R})$ and any $\sigma \in \mathrm{End}(G)$, let $\sigma_*(f)$ be the element from $\mathrm{C}^1(G;\mathbb{R})$ such that $\sigma_*(f)(\overrightarrow{ab}) = f\big(\overrightarrow{\sigma(a)\sigma(b)}\big)$ for all $ab \in \mathrm{E}(G)$. We now set

$$\mathrm{r}_G^*(f) := \frac{\sum_{\sigma\in\mathrm{End}(G)} \mathrm{r}_G\left(\sigma_*(f)\right)}{|\mathrm{End}(G)|},$$

and define $\mathrm{h}_*(G)$ to be the average value of $\mathrm{r}_G^*(f)$ for all Lipschitz cuts $f$ on $G$. Parallel to Conjecture 1.1, one may ask if the path $P_n$ is a graph which maximizes $\mathrm{h}_*(G)$ for $G \in \mathcal{G}_n$.

The height function problem introduced above has another natural generalization. For every connected graph $G$ and $x, y \in \mathrm{V}(G)$, we use $\mathrm{Dist}_G(x, y)$ to denote the distance between $x$ and $y$ in $G$. Fix an integer $k \geq 2$ and let

$\Gamma_k$ denote the infinite $k$-regular tree. Take a connected graph $G$, a vertex $v \in \mathrm{V}(G)$ and a vertex $u \in \mathrm{V}(\Gamma_k)$. Define $\mathrm{C}(G, \Gamma_k; v, u)$ to be the class of maps $f$ from $\mathrm{V}(G)$ to $\mathrm{V}(\Gamma_k)$ such that

- $f(v) = u$; and
- $\mathrm{Dist}_G(v_1, v_2) \geq \mathrm{Dist}_{\Gamma_k}(f(v_1), f(v_2))$ for all $v_1, v_2 \in \mathrm{V}(G)$.

Define the $k$-*height* of $G$, denoted by $\mathrm{h}^{(k)}(G)$, to be

$$\frac{\sum_{f \in \mathrm{C}(G,\Gamma_k;v,u)}(|\operatorname{Im}(f)| - 1)}{|\mathrm{C}(G, \Gamma_k; v, u)|},$$

where $\operatorname{Im}(f)$ means the image of the map $f$. It is not hard to see that $\mathrm{h}^{(k)}(G)$ is irrespective of the choice of $v$ and $u$ and coincides with $\mathrm{h}(G)$ when $k = 2$. Is $P_n$ the graph with maximum $k$-height among all graphs from $\mathcal{G}_n$?

In §2, we introduce our approach for tackling Conjecture 1.1 and Conjecture 1.2, present our main results (Theorem 2.5 and Theorem 2.11) and offer pertinent examples which signal our effort towards understanding several closely related objects. We prove Theorem 2.5 and Theorem 2.11 in §3 and §4, respectively.

## 2 Graph transformation and average range

To tackle Conjecture 1.1, a natural approach is to search for some suitable partial orders $<$ on $\mathcal{G}_n$ for which $G_1 < G_2$ implies $\mathrm{h}(G_1) < \mathrm{h}(G_2)$. If $P_n$ is the unique maximal element for one such partial order when restricted to some set $C \subseteq \mathcal{G}_n$, we can verify Conjecture 1.1 for $G \in C$.

Kelmans [8] introduced several operations on graphs which increase some useful graph parameter. As a variant of Kelmans's operations, Csikvári [5] proposed an operation on trees, which he called *generalized tree shift*. This operation has been thus called *KC-transformation* [3, 6] in honor of Kelmans and Csikvári. For the purpose of getting our Corollary 2.6 that verifies Conjecture 1.1 for all trees, we basically just need to use the partial order on trees generated by KC-transformations. For the possibility of extending our approach here, let us describe below a graph operation which is more general than KC-transformation.

Take a connected graph $G$ and pick $\{a, b\} \in \binom{\mathrm{V}(G)}{2}$. Let $\mathrm{V}_{a;b}(G)$ denote the set of those vertices which cannot reach $b$ without passing by $a$ in $G$.

Note that $a \in \mathrm{V}_{a;b}(G)$. Also note that $|\mathrm{V}_{a;b}(G)| > 1$ if and only if $a$ is a cut vertex of $G$. On the condition that $\min(|\mathrm{V}_{b;a}(G)|, |\mathrm{V}_{a;b}(G)|) > 1$, the *shifting operation for the pair* $(a, b)$, denoted $\mathbb{O}_{a\to b}$, can be applied on $G$ to get a new graph with one fewer cut vertex, namely $\mathbb{O}_{a\to b}(G)$, as follows: Remove the edges $bb_1, \ldots, bb_t$, where $b_1, \ldots, b_t$ are all neighbors of $b$ in $\mathrm{V}_{b;a}(G)$, and add the new edges $ab_1, \ldots, ab_t$. For technical convenience, let us think of $G$ and $\mathbb{O}_{a\to b}(G)$ as two graphs sharing the same vertex set and the same side set such that, for every $s \in \mathrm{S}(G) = \mathrm{S}(\mathbb{O}_{a\to b}(G))$,

$$\partial_{\mathbb{O}_{a\to b}(G)}(s) = \begin{cases} (a, c), & \text{if } \partial_G(s) = (b, c) \text{ and } c \in \mathrm{V}_{b;a}(G), \\ (c, a), & \text{if } \partial_G(s) = (c, b) \text{ and } c \in \mathrm{V}_{b;a}(G), \\ \partial_G(s), & \text{else.} \end{cases} \tag{2}$$

If we can obtain $G'$ from $G$ by a sequence of shifting operations, we say that $G'$ is *less* then $G$ and write it as $G' \prec G$. It is apparent that $(\mathcal{G}_n, \prec)$ is a poset for every positive integer $n$. For any graph $G$, $\mathrm{E}(G)$ can be partitioned into *blocks* where two different edges are in the same block if and only if they fall into a common simple cycle of $G$. The following easy observation hints at a reason why the shifting operation may be convenient to analyze.

**Observation 2.1.** *The two graphs $G$ and $\mathbb{O}_{a\to b}(G)$, which share the same edge set, indeed have the same block partition and so they possess the same set of Lipschitz cuts. Namely, $\mathscr{L}(G)$ and $\mathscr{L}(\mathbb{O}_{a\to b}(G))$ are the same subset of $\mathrm{C}^1(G;\mathbb{R}) = \mathrm{C}^1(\mathbb{O}_{a\to b}(G);\mathbb{R})$.*

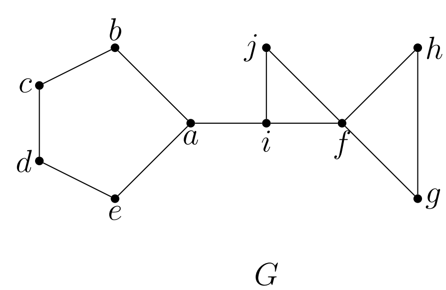
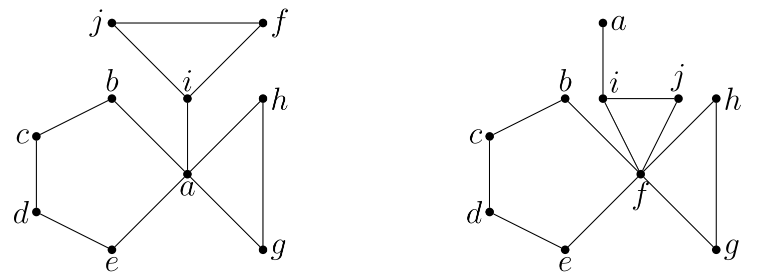



Figure 1: $\mathrm{h}(G) = \frac{17382}{7497} \approx 2.31853$.

**Example 2.2.** *We do shifting operations on the graph $G$ in Figure 1 to get two new graphs as demonstrated in Figure 2. As indicated in the figures, the height decreases after the shifting operations.*




$G' = \mathbb{O}_{a\to f}(G)$  $G'' = \mathbb{O}_{f\to a}(G)$

Figure 2: $\mathrm{h}(G') = \frac{16548}{7497} \approx 2.20728$, $\mathrm{h}(G'') = \frac{16072}{7497} \approx 2.14379$.

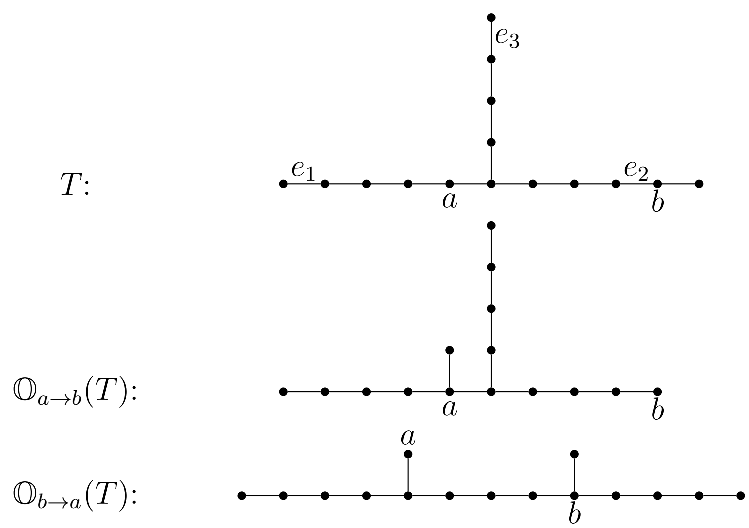



Figure 3: $\mathrm{h}(T) = \frac{18220040}{4782969} \approx 3.80936$, $\mathrm{h}(\mathbb{O}_{a\to b}(T)) = \frac{17773592}{4782969} \approx 3.71602$, $\mathrm{h}(\mathbb{O}_{b\to a}(T)) = \frac{18230214}{4782969} \approx 3.81148$.

**Example 2.3.** *In Figure 3, we demonstrate the fact that the shifting operation may increase the height. This also means that a tree with less cut vertices may have bigger height. Observe that* $\mathrm{h}(\mathbb{O}_{a\to b}(T)) + \mathrm{h}(\mathbb{O}_{b\to a}(T)) < 2\,\mathrm{h}(T)$*. Let* $T_n$ *be the graph obtained from the tree* $T$ *in Figure 3 by subdividing the edges* $e_1, e_2$ *and* $e_3$ *into paths of length* $n$*. We mention that* $\mathrm{h}(\mathbb{O}_{a\to b}(T_n)) + \mathrm{h}(\mathbb{O}_{b\to a}(T_n)) > 2\,\mathrm{h}(T_n)$ *when* $n$ *is large enough.*

**Remark 2.4.** *Let* $T$ *be a tree with two cutpoints* $a$ *and* $b$*. We do not know if it always happens* $\min\{\mathrm{h}(\mathbb{O}_{a\to b}(T)), \mathrm{h}(\mathbb{O}_{b\to a}(T))\} \le \mathrm{h}(T)$*.*

Let $G$ be a connected graph and take two different cut vertices $a$ and $b$ of $G$. We write $\mathrm{V}(G; a, b)$ for the set

$$\big(\mathrm{V}(G) \setminus (\mathrm{V}_{a;b}(G) \cup \mathrm{V}_{b;a}(G))\big) \cup \{a, b\}.$$

We have not been able to produce a theorem to explain what we saw in Example 2.2 and Example 2.3. Nevertheless, we can get the following result which is in the same spirit as these two examples. Notice that under the assumption of Theorem 2.5, $\mathbb{O}_{a\to b}(G)$ and $\mathbb{O}_{b\to a}(G)$ are isomorphic to each other.

**Theorem 2.5.** *Let* $G$ *be a connected graph and take two different cut vertices* $a$ *and* $b$ *of* $G$*. Let* $H$ *be the subgraph of* $G$ *induced by* $\mathrm{V}(G; a, b)$*. Assume that* $H$ *has an automorphism* $\sigma$ *such that* $\sigma(a) = b$ *and* $\sigma(b) = a$*. Then* $\mathrm{h}(G) > \mathrm{h}(\mathbb{O}_{a\to b}(G))$*.*

Let $T$ be a tree of order $n \ge 4$. Assume that $T$ is not the star tree $S_n$ and hence has at least two cut vertices. Pick two cut vertices $a$ and $b$ of $T$ such that $\mathrm{V}(T; a, b)$ induces a path $P$ in the tree $T$. It is obvious that $P$ has $a$ and $b$ as its endpoints and has an automorphism which swaps $a$ and $b$. Applying Theorem 2.5 now yields $\mathrm{h}(T) > \mathrm{h}(\mathbb{O}_{a\to b}(T))$. The operation $\mathbb{O}_{a\to b}$ used in this situation is just the original generalized tree shift invented by Csikvári [5]. An illustration of this so-called KC-transformation is demonstrated in Figure 4. Note that $\mathrm{E}(T) = \mathrm{E}(\mathbb{O}_{a\to b}(T))$ is the disjoint union of $E_1$, $E_2$ and $\mathrm{E}(P)$, where $E_1$ is the set of edges of $T$ whose endpoints lie in $\mathrm{V}_{a;b}(T)$ and $E_2$ is the set of edges of $T$ whose endpoints lie in $\mathrm{V}_{b;a}(T)$.

The above analysis already indicates that the star tree of order $n$ is the unique minimum element in the poset $(\mathcal{T}_n, \preceq)$. Csikvári further noticed that the path $P_n$ is the unique maximum element in the poset $(\mathcal{T}_n, \preceq)$ [5, Theorem 2.4]. Therefore, we derive from Theorem 2.5 the next result, implying that Conjecture 1.1 is valid for all trees.

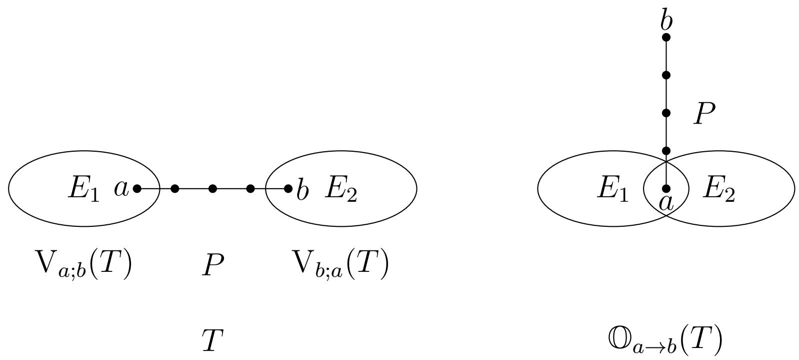



Figure 4: From $T$ to $\mathbb{O}_{a\to b}(T)$ via a KC-transformation.

**Corollary 2.6.** *Let $n$ be a positive integer and take $G \in \mathcal{T}_n$. Then,*

- $\mathrm{h}(S_n) \leq \mathrm{h}(G)$, *where equality holds if and only if* $G = S_n$;
- $\mathrm{h}(G) \leq \mathrm{h}(P_n)$, *where equality holds if and only if* $G = P_n$. $\square$

For a connected graph which is not complete, should we expect that its height will decrease after adding new edges? If that is the case, surely Conjecture 1.1 will follow from Corollary 2.6. Unfortunately, the next example says that this is not always true.

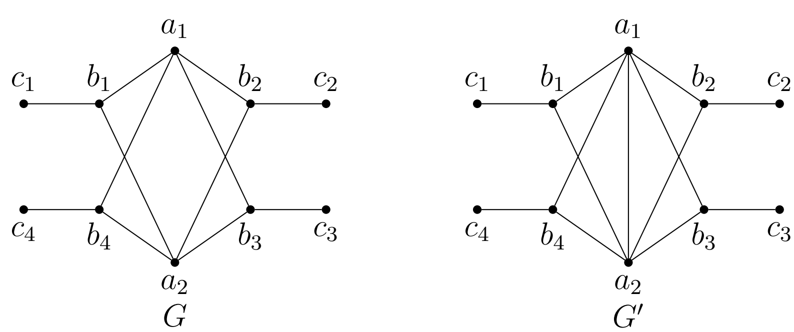



Figure 5: Adding an edge may increase the height.

**Example 2.7.** *Consider the graphs $G$ and $G'$ depicted in Figure 5, where $G'$ is obtained from $G$ by adding the edge $a_1a_2$. Computer calculations show that* $\mathrm{h}^0_{a_1,a_2}(G) = \frac{16802}{6561} \approx 2.56089 > \mathrm{h}^1_{a_1,a_2}(G) = \frac{1319}{648} \approx 2.03549 > \mathrm{h}^2_{a_1,a_2}(G) = 2$. *Note that $\mathrm{h}(G')$ is a weighted average of $\mathrm{h}^0_{a_1,a_2}(G)$ and $\mathrm{h}^1_{a_1,a_2}(G)$ while $\mathrm{h}(G)$ is* *a weighted average of* $\mathrm{h}(G')$ *and* $\mathrm{h}^2_{a_1,a_2}(G)$*. We thus know that* $\mathrm{h}(G) < \mathrm{h}(G')$*. Indeed, it holds* $\mathrm{h}(G) = \frac{22402}{9315} \approx 2.40494 < 2.41211 \approx \frac{22078}{9153} = \mathrm{h}(G')$*.*

**Example 2.8.** *Let* $n$ *and* $m$ *be two positive integers. Take the complete bipartite graph* $K_{2,m}$ *with the vertex partite sets* $\{a_1, a_2\}$ *and* $\{b_1, \ldots, b_m\}$*. Add* $m$ *new vertices* $c_1, \ldots, c_m$ *and* $m$ *new edges* $b_1c_1, \ldots, b_mc_m$ *to get a graph* $G_m$*. The graph* $G$ *in Example 2.7 (Figure 5) is simply* $G_4$*. We now do subdivision on* $G_m$ *by replacing every edge by a path of length* $n$ *to yield the graph* $G_{m;n}$ *and let* $G'_{m;n}$ *be the graph obtained from* $G_{m;n}$ *by adding the edge* $a_1a_2$*. Some simple arguments tell us that* $\lim_{m\to\infty} \mathrm{h}^0_{a_1,a_2}(G_{m;n}) = 4n > 2n = \mathrm{h}^{2n}_{a_1,a_2}(G_{m;n})$*.*

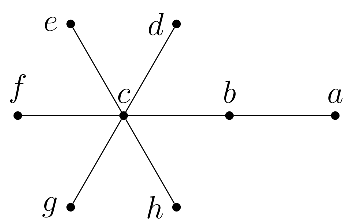



Figure 6: A tree $T$ with $\mathrm{h}^2_{b,d}(T) > \mathrm{h}(T) > \mathrm{h}^0_{b,d}(T) > \mathrm{h}^1_{b,d}(T)$.

**Example 2.9.** *For the tree* $T$ *in Figure 6, it holds* $\mathrm{h}(T) = \frac{4540}{2187} \approx 2.0759$*. Moreover,* $\mathrm{h}^0_{b,d}(T) = \frac{1460}{729} \approx 2.00274$*,* $\mathrm{h}^1_{b,d}(T) = \frac{973}{486} \approx 2.00206$ *and* $\mathrm{h}^2_{b,d}(T) = \frac{7}{3} \approx 2.33333$*.*

The above examples do not provide us any easy patterns which may direct us from Corollary 2.6 to a proof of Conjecture 1.1. Let us give a final example to suggest the possibility that every connected graph has smaller height than any of its spanning trees. Surely, if this is really true, Conjecture 1.1 follows.

**Example 2.10.** *Up to isomorphism, the graph* $G'$ *on the right of Figure 5 has in total five spanning trees, which are displayed in Figure 7. All these trees have heights bigger than the height of* $G'$*:* $\mathrm{h}(T_1) = \frac{51656}{19683} \approx 2.6244$*,* $\mathrm{h}(T_4) = \frac{52954}{19683} \approx 2.69034$*,* $\mathrm{h}(T_2) = \frac{54412}{19683} \approx 2.76442$*,* $\mathrm{h}(T_3) = \frac{55276}{19683} \approx 2.80831$*,* $\mathrm{h}(T_5) = \frac{55774}{19683} \approx 2.83361$*.*

The following result is a small step towards tackling Conjecture 1.2.

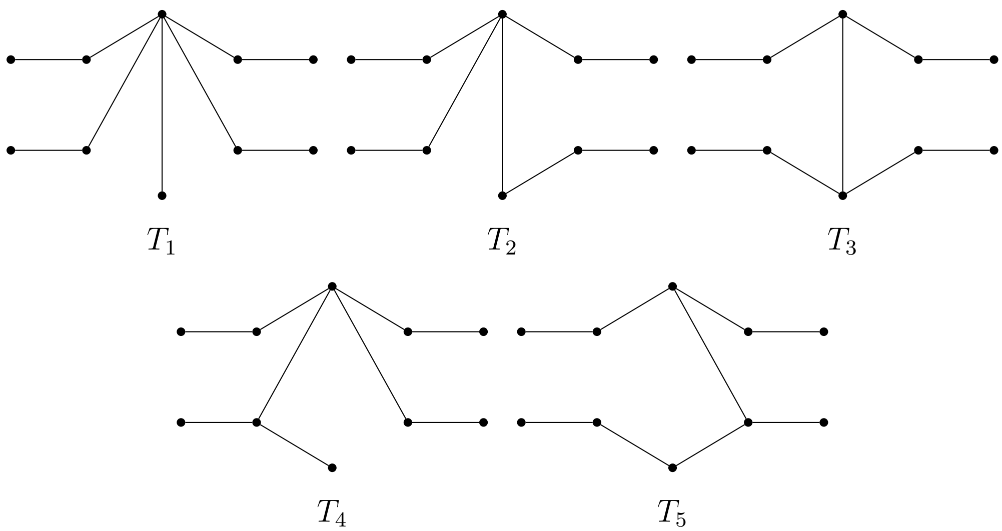



Figure 7: Five spanning trees of the graph $G'$ in Example 2.7.

**Theorem 2.11.** *Let $G$ be a connected bipartite graph and take two different cut vertices $a$ and $b$ of $G$. Let $H$ be the subgraph of $G$ induced by $\mathrm{V}(G; a, b)$. Assume that $H$ has an automorphism $\sigma$ such that $\sigma(a) = b$ and $\sigma(b) = a$. Then $\widehat{\mathrm{h}}(G) > \widehat{\mathrm{h}}(\mathbb{O}_{a\to b}(G))$.*

The argument for getting Corollary 2.6 from Theorem 2.5 can be employed to deduce the following corollary from Theorem 2.11.

**Corollary 2.12.** *Let $n$ be a positive integer and take $G \in \mathcal{T}_n$. Then,*

- $\widehat{\mathrm{h}}(S_n) \le \widehat{\mathrm{h}}(G)$, *where equality holds if and only if $G = S_n$;*
- $\widehat{\mathrm{h}}(G) \le \widehat{\mathrm{h}}(P_n)$, *where equality holds if and only if $G = P_n$.* □

## 3 Proof of Theorem 2.5

This section is devoted to a proof of Theorem 2.5. We have defined $H$ to be the subgraph of $G$ induced by $\mathrm{V}(G; a, b)$. Let us also define $H_a$ and $H_b$ to be the subgraphs of $G$ induced by $\mathrm{V}_{a;b}(G)$ and $\mathrm{V}_{b;a}(G)$, respectively. We thus get a partition of $\mathrm{S}(G)$ into three parts, $\mathrm{S}(H), \mathrm{S}(H_a)$ and $\mathrm{S}(H_b)$.

We will write $\mathbb{T}(G)$ for $\mathbb{O}_{a\to b}(G)$. We know that $\mathrm{S}(G) = \mathrm{S}(\mathbb{T}(G))$ and that $\partial_{\mathbb{T}(G)}$ and $\partial_G$ are related by Eq. (2). Also recall from Observation 2.1 that $\mathscr{L}(G)$ and $\mathscr{L}(\mathbb{T}(G))$ are equal.

We construct a map $\phi$ from $\mathrm{C}^1(H;\mathbb{R})$ to itself such that

$$\phi(f)(\overrightarrow{cd}) := -f\big(\overrightarrow{\sigma(c)\sigma(d)}\big) = f\big(\overrightarrow{\sigma(d)\sigma(c)}\big)$$

for all $f \in \mathrm{C}^1(H;\mathbb{R})$ and $\overrightarrow{cd} \in \mathrm{S}(H)$. For every $f \in \mathrm{C}^1(G;\mathbb{R})$, let $\Delta_f$ represent $\mathrm{r}_G(f) - \mathrm{r}_{\mathbb{T}(G)}(f)$ and let $\Phi(f)$ be the element of $\mathrm{C}^1(G;\mathbb{R})$ given by

$$\big(\Phi(f)|_{\mathrm{S}(H)}, \Phi(f)|_{\mathrm{S}(G)\setminus \mathrm{S}(H)}\big) := \big(\phi(f|_{\mathrm{S}(H)}), -f|_{\mathrm{S}(G)\setminus \mathrm{S}(H)}\big). \tag{3}$$

It is clear that $\Phi$ induces a bijection from $\mathscr{L}(G)$ to itself and hence

$$2\big(\mathrm{h}(G) - \mathrm{h}(\mathbb{T}(G))\big) = \sum_{f\in\mathscr{L}(G)} \big(\Delta_f + \Delta_{\Phi(f)}\big).$$

Accordingly, to prove Theorem 2.5, it suffices to show that

$$\Delta_f + \Delta_{\Phi(f)} \geq 0 \tag{4}$$

for all $f \in \mathscr{L}(G)$ and that there exists $f \in \mathscr{L}(G)$ such that (4) holds with strict inequality.

For $f \in \mathscr{L}(G)$, $A \in \{H, H_a, H_b\}$ and $v \in \mathrm{V}(G)$, we define

$$\left\{ \begin{array}{lcl} M_A^{\to v}(f) & := & \max_{x\in \mathrm{V}(A)} \int_{x\to v}^G f; \\ M_A^{v\to}(f) & := & \max_{x\in \mathrm{V}(A)} \int_{v\to x}^G f. \end{array} \right. \tag{5}$$

The shorthand notation in Eq. (5) will be of frequent use later as we have

$$\begin{aligned} \mathrm{r}_G(f) &= \max_{x,y\in \mathrm{V}(G)} \int_{x\to y}^G f \\ &= \max_{x,y\in \mathrm{V}(G)} (\int_{x\to a}^G f + \int_{a\to y}^G f) \\ &= \max_{x\in \mathrm{V}(G)} \int_{x\to a}^G f + \max_{y\in \mathrm{V}(G)} \int_{a\to y}^G f \end{aligned} \tag{6}$$

and

$$\left\{ \begin{array}{lcl} \max_{x\in \mathrm{V}(G)} \int_{x\to v}^G f & = & \max\{M_{H_a}^{\to v}(f), M_H^{\to v}(f), M_{H_b}^{\to v}(f)\}, \\ \max_{x\in \mathrm{V}(G)} \int_{v\to x}^G f & = & \max\{M_{H_a}^{v\to}(f), M_H^{v\to}(f), M_{H_b}^{v\to}(f)\}. \end{array} \right. \tag{7}$$

Let

$$\begin{cases} \Sigma_1 & := & \max\{M^{\to a}_{H_a}(f), M^{\to a}_{H}(f), M^{\to b}_{H_b}(f) - \int_{a\to b}^{G} f\}; \\ \Pi_1 & := & \max\{M^{a\to}_{H_a}(f), M^{b\to}_{H}(f), M^{b\to}_{H_b}(f) - \int_{a\to b}^{G} f\}; \\ \Sigma_2 & := & \max\{M^{\to a}_{H_a}(f), M^{\to a}_{H}(f) + \int_{a\to b}^{G} f, M^{\to b}_{H_b}(f) + \int_{a\to b}^{G} f\}; \\ \Pi_2 & := & \max\{M^{a\to}_{H_a}(f), M^{b\to}_{H}(f) + \int_{a\to b}^{G} f, M^{b\to}_{H_b}(f) + \int_{a\to b}^{G} f\}; \\ \Sigma_1' & := & \max\{M^{\to a}_{H_a}(f), M^{\to a}_{H}(f), M^{\to b}_{H_b}(f)\}; \\ \Pi_1' & := & \max\{M^{a\to}_{H_a}(f), M^{b\to}_{H}(f), M^{b\to}_{H_b}(f)\}; \\ \Sigma_2' & := & \max\{M^{\to a}_{H_a}(f), M^{\to a}_{H}(f) + \int_{a\to b}^{G} f, M^{\to b}_{H_b}(f)\}; \\ \Pi_2' & := & \max\{M^{a\to}_{H_a}(f), M^{b\to}_{H}(f) + \int_{a\to b}^{G} f, M^{b\to}_{H_b}(f)\}. \end{cases} \tag{8}$$

In view of Eqs. (6), (7) as well as the definition of $\Phi$ and $\mathbb{T}(G)$, we can write down the following formulae for $\mathrm{r}_G(f)$, $\mathrm{r}_{\mathbb{T}(G)}(f)$, $\mathrm{r}_G(\Phi(f))$ and $\mathrm{r}_{\mathbb{T}(G)}(\Phi(f))$.

$$\begin{aligned} \mathrm{r}_G(f) &= \max_{x\in \mathrm{V}(G)} \int_{x\to a}^{G} f + \max_{y\in \mathrm{V}(G)} \int_{a\to y}^{G} f \\ &= \max\{M^{\to a}_{H_a}(f), M^{\to a}_{H}(f), M^{\to a}_{H_b}(f)\} \\ &\quad + \max\{M^{a\to}_{H_a}(f), M^{a\to}_{H}(f), M^{a\to}_{H_b}(f)\} \\ &= \Sigma_1 + \Pi_2. \end{aligned} \tag{9}$$

$$\begin{aligned} \mathrm{r}_{\mathbb{T}(G)}(f) &= \max_{x\in \mathrm{V}(G)} \int_{x\to a}^{\mathbb{T}(G)} f + \max_{y\in \mathrm{V}(G)} \int_{a\to y}^{\mathbb{T}(G)} f \\ &= \max\{M^{\to a}_{H_a}(f), M^{\to a}_{H}(f), M^{\to b}_{H_b}(f)\} \\ &\quad + \max\{M^{a\to}_{H_a}(f), M^{a\to}_{H}(f), M^{b\to}_{H_b}(f)\} \\ &= \Sigma_1' + \Pi_2'. \end{aligned} \tag{10}$$

$$\begin{aligned} \mathrm{r}_G(\Phi(f)) &= \max_{x\in \mathrm{V}(G)} \int_{x\to a}^{G} \Phi(f) + \max_{y\in \mathrm{V}(G)} \int_{a\to y}^{G} \Phi(f) \\ &= \max\{M^{a\to}_{H_a}(f), M^{b\to}_{H}(f), M^{b\to}_{H_b}(f) + \int_{b\to a}^{G} f\} \\ &\quad + \max\{M^{\to a}_{H_a}(f), M^{\to b}_{H}(f), M^{\to b}_{H_b}(f) + \int_{a\to b}^{G} f\} \\ &= \Pi_1 + \Sigma_2. \end{aligned} \tag{11}$$

$$
\begin{aligned}
\mathrm{r}_{\mathbb{T}(G)}(\Phi(f)) &= \max_{x\in \mathrm{V}(G)} \int_{x\to a}^{\mathbb{T}(G)} \Phi(f) + \max_{y\in \mathrm{V}(G)} \int_{a\to y}^{\mathbb{T}(G)} \Phi(f) \\
&= \max\{M_{H_a}^{a\to}(f), M_H^{b\to}(f), M_{H_b}^{b\to}(f)\} \\
&\quad + \max\{M_{H_a}^{\to a}(f), M_H^{\to b}(f), M_{H_b}^{\to b}(f)\} \\
&= \Pi_1' + \Sigma_2'.
\end{aligned} \tag{12}
$$

Putting together Eqs. (9), (10), (11), (12) leads to

$$
\begin{aligned}
& \Delta_f + \Delta_{\Phi(f)} \\
= & \big(\mathrm{r}_G(f) - \mathrm{r}_{\mathbb{T}(G)}(f))\big) + \big(\mathrm{r}_G(\Phi(f) - \mathrm{r}_{\mathbb{T}(G)}(\Phi(f))\big) \\
= & \big((\Sigma_1 + \Sigma_2) - (\Sigma_1' + \Sigma_2')\big) + \big((\Pi_1 + \Pi_2) - (\Pi_1' + \Pi_2')\big).
\end{aligned} \tag{13}
$$

For any $x \in \mathrm{V}(H)$, the *eccentricity* of $x$ in the graph $H$ is

$$
\max_{y\in \mathrm{V}(H)} \mathrm{Dist}_H(x, y) = \max_{y\in \mathrm{V}(H)} \mathrm{Dist}_G(x, y)
$$

and will be designated by $\epsilon_H(x)$.

**Lemma 3.1.** *We can take $f \in \mathscr{L}(G)$ so that* (4) *holds as a strict inequality.*

*Proof.* Recall that $\mathrm{S}(G)$ is the disjoint union of $\mathrm{S}(H), \mathrm{S}(H_a)$ and $\mathrm{S}(H_b)$. Define the function $f$ on $\mathrm{S}(G)$ by setting

$$
f(\overrightarrow{xy}) := \begin{cases} \mathrm{Dist}_G(a, y) - \mathrm{Dist}_G(a, x), & \text{if } \overrightarrow{xy} \in \mathrm{S}(H); \\ \mathrm{Dist}_G(a, x) - \mathrm{Dist}_G(a, y), & \text{if } \overrightarrow{xy} \in \mathrm{S}(H_b), b \in \{x, y\}; \\ \mathrm{Dist}_G(a, x) - \mathrm{Dist}_G(a, y), & \text{if } \overrightarrow{xy} \in \mathrm{S}(H_a), a \in \{x, y\}; \\ 0, & \text{else.} \end{cases}
$$

It is easy to see that $f \in \mathscr{L}(G)$, as it holds $f = \mathrm{d}_G(F)$, where $F \in \mathrm{C}^0(G; \mathbb{R})$ is given by

$$
F(x) := \begin{cases} \mathrm{Dist}_G(a, x), & \text{if } x \in \mathrm{V}(G; a, b); \\ -1, & \text{if } x \in \mathrm{V}_{a;b}(G) \setminus \{a\}; \\ \mathrm{Dist}_G(a, b) - 1, & \text{if } x \in \mathrm{V}_{b;a}(G) \setminus \{b\}. \end{cases}
$$

Moreover,

$$
\left\{ \begin{array}{rcl} (M_{H_a}^{\to a}(f), M_{H_a}^{a\to}(f)) &=& (1, 0), \\ (M_H^{\to a}(f), M_H^{b\to}(f)) &=& \big(0, \epsilon_H(a) - \mathrm{Dist}_H(a, b)\big), \\ (M_{H_b}^{\to b}(f), M_{H_b}^{b\to}(f)) &=& (1, 0), \end{array} \right. \tag{14}
$$

and hence

$$\begin{cases} \Sigma_1 &= \max\{1,0,1-\mathrm{Dist}_H(a,b)\} &= 1; \\ \Sigma_2 &= \max\{1,\mathrm{Dist}_H(a,b),1+\mathrm{Dist}_H(a,b)\} &= 1+\mathrm{Dist}_H(a,b); \\ \Sigma_1' &= \max\{1,0,1\} &= 1; \\ \Sigma_2' &= \max\{1,\mathrm{Dist}_H(a,b),1\} &= \mathrm{Dist}_H(a,b); \\ \Pi_1 &= \max\{0,\epsilon_H(a)-\mathrm{Dist}_H(a,b),-\mathrm{Dist}_H(a,b)\} &= \epsilon_H(a)-\mathrm{Dist}_H(a,b); \\ \Pi_2 &= \max\{0,\epsilon_H(a),\mathrm{Dist}_H(a,b)\} &= \epsilon_H(a); \\ \Pi_1' &= \max\{0,\epsilon_H(a)-\mathrm{Dist}_H(a,b),0\} &= \epsilon_H(a)-\mathrm{Dist}_H(a,b); \\ \Pi_2' &= \max\{0,\epsilon_H(a),0\} &= \epsilon_H(a). \end{cases}$$

This combined with Eq. (13) implies $\Delta_f + \Delta_{\Phi(f)} = 1 > 0$, as was to be shown. □

After establishing Lemma 3.1, our only task is to prove (4) for all $f \in \mathscr{L}(G)$. In light of Eq. (13), we need to find a way to handle those terms in (8). It seems appropriate now to introduce some notation in max-plus algebra [4]. For any two reals $x$ and $y$, let us write $x \oplus y$ for $\max\{x,y\}$ and write $x \otimes y$ for $x+y$. For the multiplication $\otimes$ here, the multiplicative unit is 0. Hence, for any real $x$, the number $-x$ will usually be written as $x^{\otimes -1}$, as this makes $xx^{\otimes -1}$ equal to the multiplicative unit 0.

**Lemma 3.2.** *The inequality*

$$\max\{\alpha,\beta,\gamma-\delta\}+\max\{\alpha,\beta+\delta,\gamma+\delta\} \geq \max\{\alpha,\beta,\gamma\}+\max\{\alpha,\beta+\delta,\gamma\} \quad (15)$$

*holds for all real numbers $\alpha,\beta,\gamma,\delta$.*

*Proof.* We first note that

$$\begin{aligned} \alpha\otimes\gamma\otimes(\delta\oplus\delta^{\otimes-1}) - \alpha\otimes\gamma &= (\alpha\otimes\gamma)\otimes(\delta\oplus\delta^{\otimes-1})\otimes(\alpha\otimes\gamma)^{\otimes -1} \\ &= \delta\oplus\delta^{\otimes-1} \\ &\geq 0. \end{aligned} \quad (16)$$

Next, denote the left-hand side and the right-hand side of (15) by $L$ and $R$, respectively. Using some simple properties of the pair of operations $(\oplus,\otimes)$, we find that

$$\begin{aligned} L &= \big(\alpha\oplus\beta\oplus(\gamma\otimes\delta^{\otimes-1})\big)\otimes\big(\alpha\oplus(\beta\otimes\delta)\oplus(\gamma\otimes\delta)\big) \\ &= (\alpha\otimes\alpha)\oplus(\beta\otimes\alpha)\oplus(\gamma\otimes\delta^{\otimes-1}\otimes\alpha)\oplus(\alpha\otimes\beta\otimes\delta)\oplus(\beta\otimes\beta\otimes\delta) \\ &\quad \oplus(\gamma\otimes\beta)\oplus(\alpha\otimes\gamma\otimes\delta)\oplus(\beta\otimes\gamma\otimes\delta)\oplus(\gamma\otimes\gamma) \end{aligned}$$

and that

$$\begin{aligned} R &= \big(\alpha \oplus \beta \oplus \gamma\big) \otimes \big(\alpha \oplus (\beta \otimes \delta) \oplus \gamma\big) \\ &= (\alpha \otimes \alpha) \oplus (\beta \otimes \alpha) \oplus (\gamma \otimes \alpha) \oplus (\alpha \otimes \beta \otimes \delta) \oplus (\beta \otimes \beta \otimes \delta) \oplus (\gamma \otimes \beta \otimes \delta) \\ &\quad \oplus (\alpha \otimes \gamma) \oplus (\beta \otimes \gamma) \oplus (\gamma \otimes \gamma). \end{aligned}$$

By pairing off identical terms from the above expressions of $L$ and $R$, it follows from (16) that $L \geq R$, as wanted. □

Taking

$$(\alpha, \beta, \gamma, \delta) := \big(M_{H_a}^{\to a}(f), M_H^{\to a}(f), M_{H_b}^{\to b}(f), \int_{a\to b}^{G} f\big)$$

in Lemma 3.2 gives

$$\Sigma_1 + \Sigma_2 \geq \Sigma_1' + \Sigma_2';$$

while taking

$$(\alpha, \beta, \gamma, \delta) := \big(M_{H_a}^{a\to}(f), M_H^{b\to}(f), M_{H_b}^{b\to}(f), \int_{a\to b}^{G} f\big)$$

in Lemma 3.2 yields

$$\Pi_1 + \Pi_2 \geq \Pi_1' + \Pi_2'.$$

By now, (4) is direct from Eq. (13). This completes the proof of Theorem 2.5.

## 4 Proof of Theorem 2.11

In §3, we studied the average range of $\mathrm{C}^1_{\mathcal{I}_G}(G;\mathbb{R})$ for the constraint $\mathcal{I}_G$ listed in Eq. (1). But, checking the proof given in §3 shows that the analysis there can be applied for much more general constraints $I$. In particular, let us fix $G$ to be a connected bipartite graph and present below a proof of Theorem 2.11.

Firstly, we check that $\widehat{\mathscr{L}}(G) = \widehat{\mathscr{L}}(\mathbb{T}(G))$ while the map $\Phi$ defined in Eq. (3) gives rise to a bijection from $\widehat{\mathscr{L}}(G)$ to itself. It follows that

$$2(\widehat{\mathrm{h}}(G) - \widehat{\mathrm{h}}(\mathbb{T}(G)) = \sum_{f \in \widehat{\mathscr{L}}(G)} (\Delta_f + \Delta_{\Phi(f)}).$$

As before, it then suffices to prove the inequality $\Delta_f + \Delta_{\Phi(f)} \geq 0$ for all $f \in \widehat{\mathscr{L}}(G)$ and to justify the existence of $f \in \widehat{\mathscr{L}}(G)$ satisfying

$$\Delta_f + \Delta_{\Phi(f)} > 0. \tag{17}$$

Replacing everywhere $f \in \mathscr{L}(G)$ by $f \in \widehat{\mathscr{L}}(G)$, we can set up the inequality in exactly the same way as we prove (4) for $f \in \mathscr{L}(G)$. To show the existence of $f \in \widehat{\mathscr{L}}(G)$ for which (17) holds, we need to establish a result for $\widehat{\mathscr{L}}(G)$ parallel to Lemma 3.1. Note that the proof for Lemma 3.1 does not directly work for the case of $\widehat{\mathscr{L}}(G)$. But, adjusting a little bit of the argument there still allows us to enunciate the following lemma, which concludes the proof of Theorem 2.11.

**Lemma 4.1.** *Let $G$ be a connected bipartite graph. Then we can always find $f \in \widehat{\mathscr{L}}(G)$ to fulfil* (17)*.*

*Proof.* Take $F \in \mathrm{C}^0(G;\mathbb{R})$ so that

$$F(x) = \begin{cases} \mathrm{Dist}_G(a,x), & \text{if } x \in \mathrm{V}(G;a,b); \\ -1, & \text{if } x \in \mathrm{V}_{a;b}(G) \text{ and } \mathrm{Dist}_G(x,a) \text{ is odd;} \\ 0, & \text{if } x \in \mathrm{V}_{a;b}(G) \text{ and } \mathrm{Dist}_G(x,a) \text{ is even;} \\ \mathrm{Dist}_G(a,b)-1, & \text{if } x \in \mathrm{V}_{b;a}(G) \text{ and } \mathrm{Dist}_G(x,b) \text{ is odd;} \\ \mathrm{Dist}_G(a,b), & \text{if } x \in \mathrm{V}_{b;a}(G) \text{ and } \mathrm{Dist}_G(x,b) \text{ is even.} \end{cases}$$

Setting $f := \mathrm{d}_G(F)$, we can see that $f$ falls into $\widehat{\mathscr{L}}(G)$. In addition, we check that Eq. (14) still holds for this $f$ and so the same computation process as in Lemma 3.1 shows that $\Delta_f + \Delta_{\Phi(f)} = 1 > 0$. □

## References


[1] I. Benjamini, O. Häggström, E. Mossel, On random graph homomorphisms into $\mathbb{Z}$, Journal of Combinatorial Theory, Series B 78 (2000) 86–114.

[2] I. Benjamini, Y. Peres, Tree-indexed random walks on groups and first passage percolation, Probability Theory and Related Fields 98 (1994) 91–112.

[3] B. Bollobás, M. Tyomkyn, Walks and paths in trees, Journal of Graph Theory 70 (2012) 54–66.

[4] P. Butkovič, Max-linear Systems: Theory and Algorithms, Springer, 2010.

[5] P. Csikvári, On a poset of trees, Combinatorica 30 (2010) 125–137.

[6] P. Csikvári, Z. Lin, Graph homomorphisms between trees, Electronic Journal of Combinatorics 21 (2014) Paper #P4.9, 38 pp.

[7] A. Erschler, Random mappings of scaled graphs, Probability Theory and Related Fields 144 (2009) 543–579.

[8] A. K. Kelmans, On graphs with randomly deleted edges, Acta Mathematica Hungarica 37 (1981) 77–88.

[9] M. Loebl, J. Nešetřil, B. Reed, A note on random homomorphism from arbitrary graphs to $\mathbb{Z}$, Discrete Mathematics 273 (2003) 173–181.

[10] R. Peled, W. Samotijy, A. Yehudayoff, Grounded Lipschitz functions on trees are typically flat, Electronic Communications in Probability 18 (2013) 1–9.

[11] R. Peled, W. Samotijy, A. Yehudayoff, Lipschitz functions on expanders are typically flat, Combinatorics, Probability and Computing 22 (2013) 566–591.