A Glimpse at Boolean Linear Dynamical Systems

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A dynamical system on digraph

Figure: $\Gamma$
A dynamical system on digraph

Figure: $\Gamma$

\{2, 5\}
A dynamical system on digraph

\[ \{2, 5\} \rightarrow \{3\} \]

Figure: $\Gamma$
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Figure: $\Gamma$

$\{2, 5\} \rightarrow \{3\} \rightarrow \{4\}$
A dynamical system on digraph

Figure: $\Gamma$

$\{2, 5\} \rightarrow \{3\} \rightarrow \{4\} \rightarrow \{1, 5\}$
A dynamical system on digraph

{2, 5} → {3} → {4} → {1, 5} → {2, 3}

Figure: \(\Gamma\)
A dynamical system on digraph

{2, 5} → {3} → {4} → {1, 5} → {2, 3} → {3, 4}

Figure: Γ
A dynamical system on digraph

Figure: $\Gamma$

\{2, 5\} $\rightarrow$ \{3\} $\rightarrow$ \{4\} $\rightarrow$ \{1, 5\} $\rightarrow$ \{2, 3\} $\rightarrow$ \{3, 4\} $\rightarrow$ \{1, 4, 5\}
A dynamical system on digraph

Figure: \( \Gamma \)

\{2, 5\} \rightarrow \{3\} \rightarrow \{4\} \rightarrow \{1, 5\} \rightarrow \{2, 3\} \rightarrow \{3, 4\} \rightarrow \{1, 4, 5\} \rightarrow \{1, 2, 3, 5\}
Phase space

Figure: Phase space of $\Gamma$
Phase space

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Boolean linear dynamical systems

Let $k$ be a positive integer and $\mathcal{P}_k = \{1, 2, \ldots, k\}$. Let \( \text{Set}_k \) denote \( 2^{\mathcal{P}_k} \setminus \{\emptyset\} \).

A map $f$ from \( \text{Set}_k \) to \( \text{Set}_k \) is essential provided $f(A) \cup f(B) = f(A \cup B)$, and $f(\mathcal{P}_k) = \mathcal{P}_k$.

Essential maps are just digraphs without sinks and sources, or just Boolean matrices without zero lines.

Let $F$ be a set of essential maps on \( \text{Set}_k \), the iterations of elements of $F$ giving the dynamics of the system. We call \( (\text{Set}_k, F) \) a Boolean linear dynamical system.
Let $k$ be a positive integer and $[k] = \{1, 2, \ldots, k\}$. Let $\text{Set}_k$ denote $2^\[k\] \setminus \{\emptyset\}$. A map $f$ from $\text{Set}_k$ to $\text{Set}_k$ is essential provided

- $f(A) \cup f(B) = f(A \cup B)$, and
- $f([k]) = [k]$
Boolean linear dynamical systems

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Let $\mathcal{F}$ be a set of essential maps on $\text{Set}_k$, the iterations of elements of $\mathcal{F}$ giving the dynamics of the system. We call $(\text{Set}_k, \mathcal{F})$ a **Boolean linear dynamical system**.
Longest path in a phase space

The phase space of Boolean linear dynamical system \((\text{Set}_k, F)\) denoted by \(\text{PS}_F\), is the digraph with vertex set \(\text{Set}_k\) and arc set \(\{s \rightarrow f(s) : s \subset \{1, \ldots, k\}, f \in F\}\).

We use \(g(F)\) to denote the length of longest path in \(\text{PS}_F\), and \(g(A \rightarrow B)\) to denote the length of longest path in \(\text{PS}_F\) from \(A\) to \(B\) for any \(A, B \in \text{Set}_k\).

The Boolean linear dynamical system \((\text{Set}_k, F)\) is primitive provided every long enough walk in \(\text{PS}_F\) will reach \(\{1, \ldots, k\}\).

If \(F\) is primitive, we say \(g(F)\) is the primitive index of \(F\).
The phase space of Boolean linear dynamical system \((\text{Set}_k, \mathcal{F})\) denoted by \(\mathcal{PS}_\mathcal{F}\), is the digraph with vertex set \(\text{Set}_k\) and arc set 
\[\{s \rightarrow f(s) : s \subset [k], f \in \mathcal{F}\}\]
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We use $g(\mathcal{F})$ to denote the length of longest path in $\mathcal{PS}_\mathcal{F}$, and $g(\mathcal{F})_{A \to B}$ to denote the length of longest path in $\mathcal{PS}_\mathcal{F}$ from $A$ to $B$ for any $A, B \in \text{Set}_k$. 
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Reachability

Theorem (Coxson, Laroson, Schneider)

Let $\Gamma$ be a digraph with $k$ vertices. If $|B| = 1$, then $g(\{\Gamma\})_{A \rightarrow B} \leq k$. 

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Theorem (Wu, Xu, Z.)
Let $\Gamma$ be a digraph with diameter $D_{\Gamma}$. Then for any $A, B \in \text{Set}_k$, $g(\{\Gamma\})_{A \to B} \leq |B|D_{\Gamma}$. If $\Gamma$ is primitive, then for any $A \in \text{Set}_k$ and any positive integer $s$, $|\Gamma^s A| \geq \frac{s}{D_{\Gamma}}$. 
Absolute upper bound of the primitive index

Let $F$ be a primitive set of essential maps on $\text{Set}^k$ with $g(F) = 2^k - 2$. That means there exists a unique path in $\text{PS}_F$ with length $2^k - 2$. This path induces a total order $\pi_F$ on $\text{Set}^k$.

Figure: $F$ and $\text{PS}_F$
Absolute upper bound of the primitive index

Let $\mathcal{F}$ be a primitive set of essential maps on $\text{Set}_k$ with $g(\mathcal{F}) = 2^k - 2$. That means there exists a unique path in $\mathcal{P}S_{\mathcal{F}}$ with length $2^k - 2$. This path induces a total order $\pi_{\mathcal{F}}$ on $\text{Set}_k$. 
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![Diagram of paths and order]
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Figure: $\mathcal{F}$ and $\mathcal{PS}_\mathcal{F}$
The broken Boolean lattice, denoted $B_k$, is ordered by inclusion.

$\{1, 2, 3\}$

$\{1, 2\}$

$\{1, 3\}$

$\{2, 3\}$

$\{1\}$

$\{2\}$

$\{3\}$

**Figure:** $B_3$

**Lemma (Wu, Z.)**

Let $\pi$ be a total order on $\text{Set}_k$. There exist a set of essential maps $F$ such that $g(F) = 2^k - 2$ and $\pi F = \pi$ if and only if $\pi$ is a linear extension of $B_k$. 
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\[
\begin{array}{c}
\{1, 2, 3\} \\
| \\
\{1, 2\} & \{1, 3\} & \{2, 3\} \\
| & \times & \times \\
\{1\} & \{2\} & \{3\}
\end{array}
\]

**Figure:** $B_3$
Broken Boolean lattice

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![Diagram of $B_3$]

**Figure: $B_3$**

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Let $\pi$ be a total order on $\text{Set}_k$. There exist a set of essential maps $\mathcal{F}$ such that $g(\mathcal{F}) = 2^k - 2$ and $\pi_{\mathcal{F}} = \pi$ if and only if $\pi$ is a linear extension of $B_k$. 
Definition

\[ \gamma_{k,\pi} = \min\{|F| : g(F) = 2^k - 2, \pi_F = \pi\}. \]
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Theorem (Wu, Z.)

Let \( \pi \) be the lexicographical order on \( \text{Set}_k \). Then \( \gamma_k \leq \gamma_{k,\pi} = k \).
Some problems

\[ \gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 3, \gamma_5 \in \{2, 3, 4\}, \gamma_6 = \ldots \]

Is it true that \( \gamma_4 \leq \gamma_5 \leq \gamma_6 \leq \ldots \) ?

Nontrivial lower bound for \( \gamma_k \)?

Which kind of total orders \( \pi \) on \( \text{Set}^k \) give us a small \( \gamma_k, \pi \)?

Let \( F = \{ \Gamma_1, \Gamma_2, \ldots, \Gamma_r \} \) be a primitive set of essential maps on \( \text{Set}^k \) and \( A, B \in \text{Set}^k \) in \( \text{PS}_F \). Can we estimate \( g(F) \)?

\[ g(F) \leq |B|^{D_{\Gamma_1}D_{\Gamma_2} \ldots D_{\Gamma_r}} \]

Thank you!
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