A Glimpse at Boolean Linear Dynamical Systems

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Joint with Yaokun Wu & Zeying Xu

A dynamical system on digraph

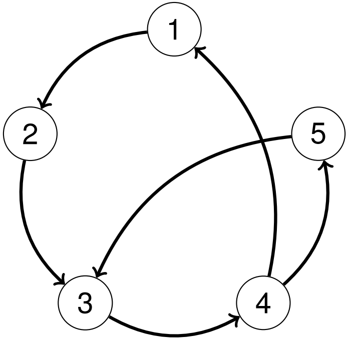
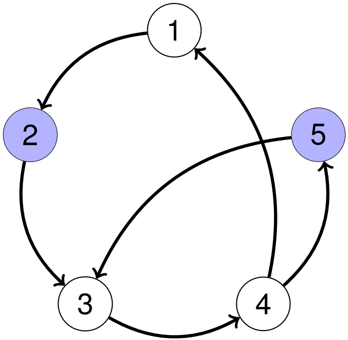


Figure: Γ

A dynamical system on digraph



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$\{2, 5\}$

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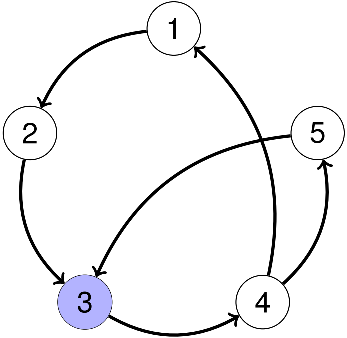

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$\{2, 5\} \to \{3\}$

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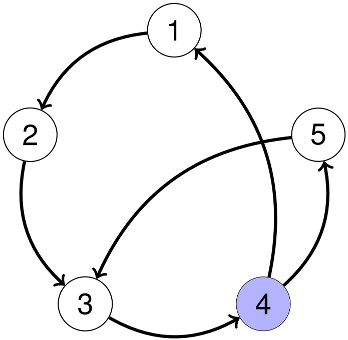

Figure: Γ

$\{2, 5\} \to \{3\} \to \{4\}$

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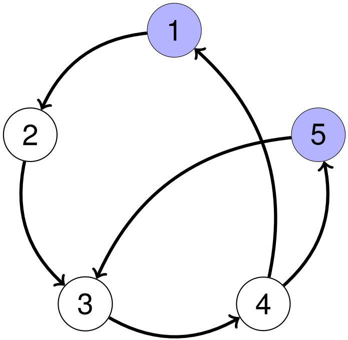

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$\{2,5\} \to \{3\} \to \{4\} \to \{1,5\}$

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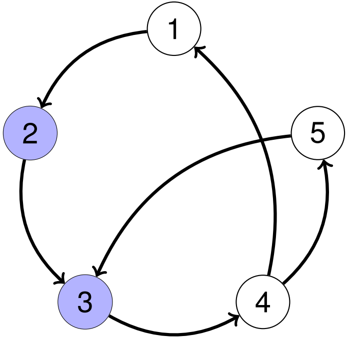

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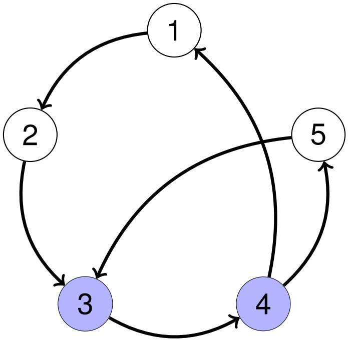

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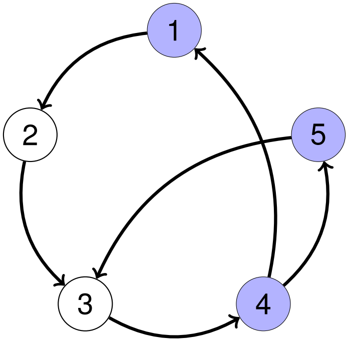

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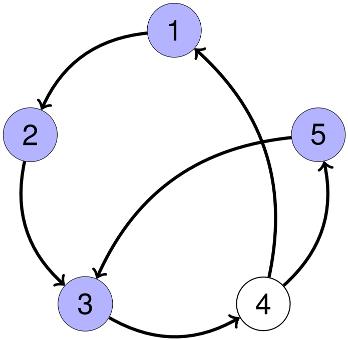

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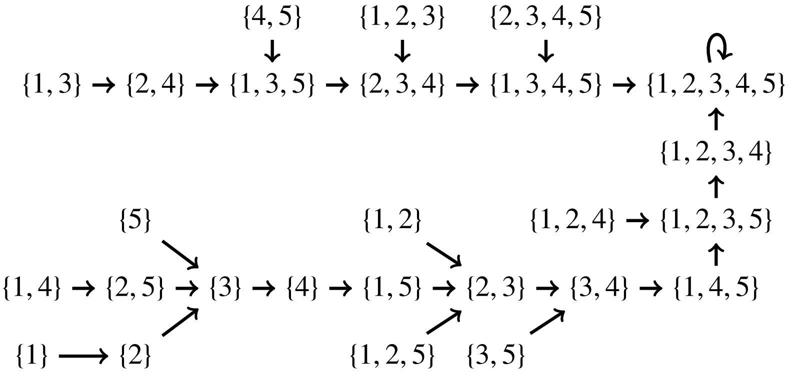
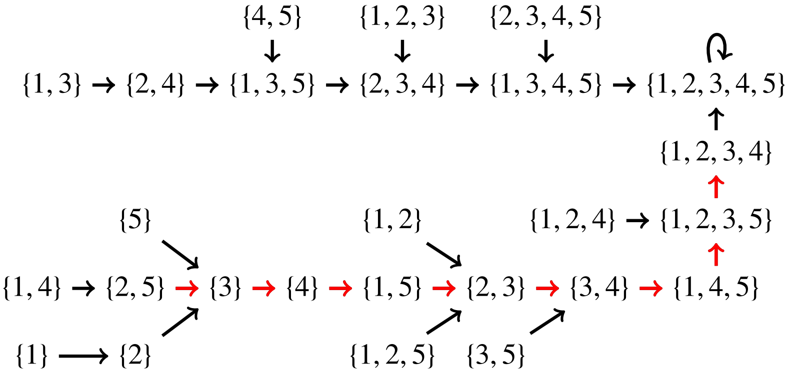


Figure: Phase space of Γ

Phase space



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Let $\mathcal{F}$ be a set of essential maps on Set_k, the iterations of elements of $\mathcal{F}$ giving the dynamics of the system. We call $(\mathrm{Set}_k, \mathcal{F})$ a Boolean linear dynamical system.

Longest path in a phase space

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The phase space of Boolean linear dynamical system $(\mathrm{Set}_k, \mathcal{F})$ denoted by $\mathcal{PS}_{\mathcal{F}}$, is the digraph with vertex set Set_k and arc set $\{s \to f(s) : s \subset [k], f \in \mathcal{F}\}$

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We use $g(\mathcal{F})$ to denote the length of longest path in $\mathcal{PS}_{\mathcal{F}}$, and $g(\mathcal{F})_{A \to B}$ to denote the length of longest path in $\mathcal{PS}_{\mathcal{F}}$ from A to B for any $A, B \in \mathrm{Set}_k$.

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If $\mathcal{F}$ is primitive, we say $g(\mathcal{F})$ is the primitive index of $\mathcal{F}$.

Reachability

Theorem (Coxson, Laroson, Schneider)

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Theorem (Wu, Xu, Z.)

Let Γ be a digraph with diameter D_Γ. Then for any $A, B \in \mathrm{Set}_k$, $g(\{\Gamma\})_{A\to B} \le |B| D_\Gamma$. If Γ is primitive, then for any $A \in \mathrm{Set}_k$ and any positive integer s, $|\Gamma^s A| \ge \frac{s}{D_\Gamma}$.

Absolute upper bound of the primitive index

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Let $\mathcal{F}$ be a primitive set of essential maps on Set_k with $g(\mathcal{F}) = 2^k - 2$. That means there exists a unique path in $\mathcal{PS}_{\mathcal{F}}$ with length $2^k - 2$. This path induces a total order $\pi_{\mathcal{F}}$ on Set_k.

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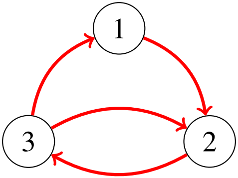

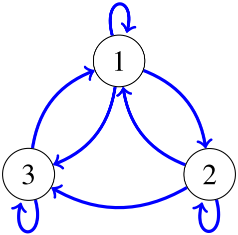

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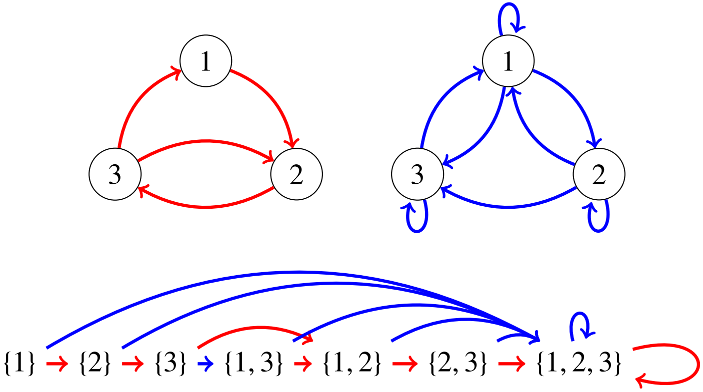

Figure: $\mathcal{F}$ and $\mathcal{PS}_{\mathcal{F}}$

Broken Boolean lattice

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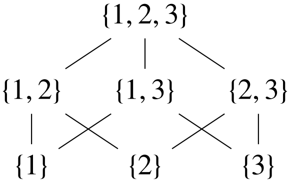
The broken Boolean lattice, denoted B_k, is Set_k ordered by inclusion.



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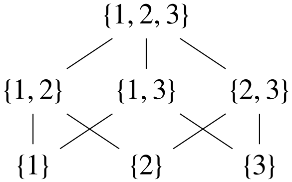

Figure: B_3

Lemma (Wu, Z.)

Let π be a total order on Set_k. There exist a set of essential maps $\mathcal{F}$ such that $g(\mathcal{F}) = 2^k - 2$ and $\pi_{\mathcal{F}} = \pi$ if and only if π is a linear extension of B_k.

Definition

- $\gamma_{k,\pi} = \min\{|\mathcal{F}| : g(\mathcal{F}) = 2^k - 2, \pi_{\mathcal{F}} = \pi\}$.

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Theorem (Cohen, Seller)

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Theorem (Wu, Z.)

Let π be the lexicographical order on Set_k. Then $\gamma_k \leq \gamma_{k,\pi} = k$.

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Thank you!