Road coloring problem for completely reachability

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Ural Federal University

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Deterministic finite automata

A deterministic finite automaton (DFA) is a triple $\mathcal{A} = (Q, \Sigma, \delta)$ where

- Q is a finite set, called the state set;
- Σ is a finite set, called the input alphabet;
- $\delta : Q \times \Sigma \to Q$ is a map, called the transition function.

Σ^* stands for the set of all words over Σ including the empty word ϵ. The function δ extends to a function $Q \times \Sigma^* \to Q$ (still denoted by δ) via the following recursion: For every $q \in Q$, we set

$$\delta(q, \epsilon) = q$$
$$\delta(q, wa) = \delta(\delta(q, w), a)$$

for all $w \in \Sigma^*$ and $a \in \Sigma$.
To simplify the notation, we often write $q.w$ for $\delta(q, w)$ and $P.w$ for $\{\delta(q, w) : q \in P\}$, where $P \subseteq Q$.

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Completely reachable automata

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be a DFA.

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- A DFA is completely reachable if every non-empty subset of its states is reachable.
- A DFA is synchronizing if there exists a reachable singleton set $\{x\} \subseteq Q$.

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Digraph

A digraph is a quadruple $G = (V, E, i, t)$ where V, E are non-empty sets and $i, t : E \to V$.

- The elements in V are called vertices;
- the elements of E are called edges;

for an edge $e \in E$,

- the vertex $i(e)$ is called the initial vertex of e;
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Neighbours and degrees

Let v be a vertex in a digraph G.

- The out-neighbour of v is the set $\{i(e) : t(e) = v, e \in E\}$, denoted $N_+(v)$.
- The in-neighbour of v is the set $\{t(e) : i(e) = v, e \in E\}$, denoted $N_-(v)$.
- For a subset $U \in V$,
  - write $N_+(U)$ for the set $\{u : N_+(u), u \in U\}$;
  - write $N_-(U)$ for the set $\{u : N_-(u), u \in U\}$.
- The out-degree of v is the number of edges whose initial vertex is v, denoted $\mathrm{d}_+(v)$.
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Road colorings

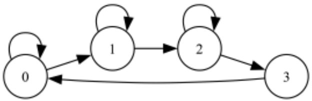
For a set X, write $\mathcal{P}(X)$ for the power set of X.

A road coloring of a finite digraph $G = (V, E, i, t)$ is a function $\alpha : E \to \mathcal{P}(\Sigma) \setminus \{\emptyset\}$ such that for every vertex $v \in V$, the family of sets

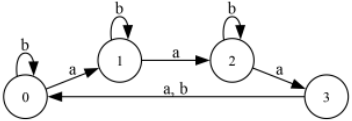
$$\{\alpha(e) : t(e) = v, e \in E\}$$

forms a partition of Σ.

Example



a digraph



a digraph with a road coloring

x	0	1	2	3
x.a	1	2	3	0
x.b	0	1	2	0

an automaton

Automata from road colorings

Let $\alpha : E \to \mathcal{P}(\Sigma) \setminus \{\emptyset\}$ be a road coloring of G.
Define $\mathcal{A}(G, \alpha)$ to the automaton (V, Σ, δ) such that for every $v \in V$ and $a \in \Sigma$,

$$v.a = t(e)$$

where e is the edge such that $i(e) = v$ and $a \in \alpha(e)$.
The road coloring α

- is called a sychronizing coloring if $\mathcal{A}(G, \alpha)$ is a synchronizing automaton;
- is called a completely reachable coloring if $\mathcal{A}(G, \alpha)$ is a completely reachable automaton.

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Road coloring problem

Problem

For a given digraph G, how to determine whether or not G admits a road coloring such that the correponding automaton fulfills some properties (i.e., synchronizing, completely reachable, ...)?

Trahtman's Road Coloring Theorem

- The period of a strongly connected digraph G is the greatest common divisor of the lengths of its cycles, denoted $p(G)$.
- A digraph is called aperiodic if its period equals 1.

Theorem (Trahtman[1], 2009)

Let $G = (V, E, i, t)$ be a strongly connected digraph and $d = \max\{\mathrm{d}_+(v), v \in V\}$. The following are equivalent.

1. *The digraph G admits a synchronizing coloring.*
2. *The digraph G admits a synchronizing coloring with d colors.*
3. *The digraph G is aperiodic.*

[1] A. N. Trahtman (2009). "The road coloring problem". In: *Israel J. Math.* 172, pp. 51–60. ISSN: 0021-2172,1565-8511.

Completely reachable colorings

Theorem (Z., 2023+)

A digraph $G = (V, E, i, t)$ admits a completely reachable coloring if and only if

1. *G is strongly connected,*
2. *G is aperiodic,*
3. *for every subset $U \subseteq V$, $|U| \leq |N_{-}(U)|$.*

Theorem (Z., 2023+)

Let $k \geq 2$ be a fixed integer. To determine a given digraph $G = (V, E, i, t)$ whether or not it admits a completely reachable with k colors is NP-complete.

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Cyclic Decomposition Theorem

Theorem (Cyclic Decomposition Theorem)

Let G be a strongly connected digraph of period p. The vertex set can be partition into p sets $\{C_i : i \in \mathbb{Z}_p\}$ such that $N_+(C_i) = C_{i+1}$ for every $i \in \mathbb{Z}_p$.
Moreover, for each vertex $v \in C_j$ for some j, there exists a positive integer k such that

$$N_-^k(v) = \underbrace{N_-(\cdots N_-(}_{k} v)) = C_j.$$

Hall's Marriage Theorem

A bipartite graph $H = (X, Y, E)$ is a triple, where X, Y are two nonempty sets and $E \subseteq X \times Y$.

The elements in $X \cup Y$ are vertices and the elements in E are edges. A X-perfect matching of H is a matching, a set of disjoint edges, which covers every vertex in X. For $U \subseteq X$, the neighborhood of U is the set $\{w : (u, w) \in E, u \in U\}$, denoted $N(U)$.

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Let $H = (X, Y, E)$ be a bipartite graph. There exists an X-perfect matching if and only if for every subset $U \subseteq X$, we have $|U| \leq N(U)$.

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A digraph $G = (V, E, i, t)$ admits a completely reachable coloring if and only if

1. *G is strongly connected,*
2. *G is aperiodic,*
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"$\Rightarrow$":

- Let α be a completely reachable coloring of G and its color set is Σ. The corresponding autoaton is $\mathcal{A}(G,\alpha) = (V,\Sigma,\delta)$.
- For every two vertices u, v, by completely reachability, there exists a word w such that $V.w = \{u\}$ and then $\delta(v,w) = u$. Then there exists a walk in G from v to u. This implies that G is strongly connected
- By Cyclic Decomposition Theorem, the vertex set V can be partitioned into p sets $\{C_i : i \in \mathbb{Z}_p\}$ such that $N_+(C_i) = C_{i+1}$ for every $i \in \mathbb{Z}_p$.
- Then for any word $w \in \Sigma^*$ and $i \in \mathbb{Z}_p$, $V.w \cap C_i \neq \emptyset$. Since every singleton set is reachable. we have $p = 1$ and then G is aperiodic.
- For a non-empty subset $U \subseteq V$, take a word $w = w'a \in \Sigma^*$ such that $V.w = U$. Let W be the set $V.w'$. Then

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The proof, cont'd

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- Define H to be the bipartite graph $H = (V_1, V_2, E_H)$ such that $V_1 = V_2 = V$ and $(u, v) \in E_H$ if there exists $e \in E$ such that $t(e) = u$ and $i(e) = v$.
- Observe that for every non-empty subset $U \subseteq V_1$, then $|U| \leq |N(U)|$.

Let W be a non-empty subset of V_1. Let H' be the induced subgraph of H on $W \cup N(W)$. By the Hall's Marriage Theorem, there exists a W-perfect matching M in H'.

Now we can define a function $f_W : V_2 \to V_1$ as following:

1. for $y \in V_2$ which is covered by edge $(x, y) \in M$, set $f_W(y) = x$;
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- Define H to be the bipartite graph $H = (V_1, V_2, E_H)$ such that $V_1 = V_2 = V$ and $(u, v) \in E_H$ if there exists $e \in E$ such that $t(e) = u$ and $i(e) = v$.
- Observe that for every non-empty subset $U \subseteq V_1$, then $|U| \leq |N(U)|$.

Let W be a non-empty subset of V_1. Let H' be the induced subgraph of H on $W \cup N(W)$. By the Hall's Marriage Theorem, there exists a W-perfect matching M in H'.

Now we can define a function $f_W : V_2 \to V_1$ as following:

1. for $y \in V_2$ which is covered by edge $(x, y) \in M$, set $f_W(y) = x$;
2. for $y \in N(W)$ which is not covered by the matching M, set $f_W(y)$ to be an arbitrary vertex in $W \cap N(y)$.
3. for $y \in V_2 \setminus N(W)$, set $f_W(y)$ to be an arbitrary vertex in $N(y)$.

It is clear that $W = f_W(N(W))$.

The proof, cont'd

Now we construct a road coloring $\alpha : E \to \mathcal{P}(\Sigma)$, where $\Sigma = \mathcal{P}(V) \setminus \{\emptyset\}$ by setting

$$\alpha(e) = \{U : f_U(t(e)) = i(e), \emptyset \neq U \subseteq V\}.$$

Let $\mathcal{A} = (V, \Sigma, \delta) = \mathcal{A}(G, \alpha)$. Note that for every non-empty subset U, we have

$$\delta(N_-(U), U) = U.$$

- Let U_0 be an arbitrary non-empty subset of V, define $U_i = N_-(U_{i-1})$ for all positive integer i.
- Since G is strongly connected and aperiodic, by Cyclic Decomposition Theorem, there exists an integer k such that $U_k = V$.
- Then $U_0 = \delta(V, U_{k-1}U_{k-2}\cdots U_1U_0)$ is reachable.
- Hence $\mathcal{A}$ is completely reachable and G admits a completely reachable coloring.

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Theorem (Z., 2023+)

Let $k \geq 2$ be a fixed integer. To determine a given digraph $G = (V, E, i, t)$ whether or not it admits a completely reachable with k colors is NP-complete.

Proof for $k = 2$:

- Let $G = (V, E, i, t)$ be a digraph such that
  1. $|V|$ is an odd prime number;
  2. for every vertex v, $d_+(v) = 2$;
  3. there exist vertices x and y such that $d_-(x) = 1$, $d_-(y) = 3$ and $d_-(z) = 2$, for each $z \in V \setminus \{x, y\}$.
- Then G admits a completely reachable coloring if and only if G has a hamitonian cycle (a directed cycle visits each vertex once).
- To determine whether or not such a given graph G has a hamitonian cycle is NP-complelte. (Our proof is obtained from the proof in Plesnik's paper[2] with some small modification.)

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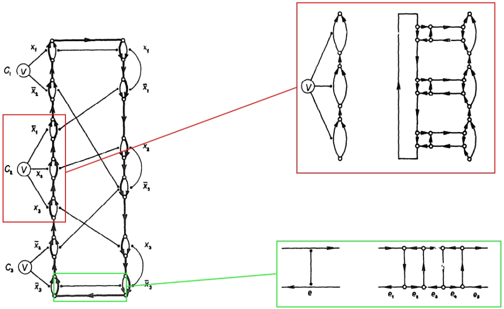
Plesnik's Reduction

- Reduce from the 3-SAT problem.
- Construct a digraph G for a given boolean formula F. Vertices of G have in-degree and out-degree at most 2.
- G has a hamitonian cycle if and only if F is satisfiable.
- The following figure is the digraph corresponding to $F = (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_2} \vee \overline{x_3})$.



Open problems

Let $f : \mathbb{N} \to \mathbb{N}$.

1. For a given digarph G with n vertices, is there a polynomial-time algorithm to determine whether G admits a completely reachable coloring which uses $f(n)$ colors?
2. For a given digraph G which admits a completely reachable coloring, can we find one completely reachable coloring in polynomial time?

Thank you

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