Three primitivities on matrix tuples

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Convension

In this talk, a "matrix" means a non-negative (Boolean) square matrix.

non-negative matrix tuple $\leftrightarrow$ Boolean matrix tuple $\leftrightarrow$ arc-labelled digraph

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow$$

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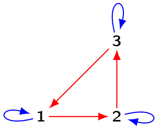

Perron-Frobenius-Romanovsky theorem

A nonnegative n-by-n matrix A is called **primitive** if $A^k > 0$ (entrywise) for some $k \geq 0$.

Theorem (Perron-Frobenius, 1912; Romanovsky, 1933)

An irreducible matrix A is not primitive if one of the following equivalent conditions is satisfied:

1. *the length of all cycles of the digraph of the matrix A have greatest common divisor $r > 1$.*
2. *there is a partition of the set $\{1, \ldots, n\}$ into $r > 1$ sets $\pi = (V_1, \ldots, V_r)$ such that A is a block permutation matrix with respect to π.*

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Primitive index

- Suppose that we wish to decide whether or not a nonnegative matrix A is primitive by computing the sequence of powers $A, A^2, A^3, \ldots$ (although this may not a clever way). It would be nice to know when we have computed enough powers of A to render a judgement.
- The minimal positive integer m such that $A^m > 0$ is called the **primitive index** of A, denoted by $\mathrm{p}(A)$.
- Define $\mathrm{p}(n) := \max\{\mathrm{p}(A) : A \text{ is a primitive } n\text{-by-}n \text{ matrix}\}$.

Theorem (Wielandt[1], 1959)

If A is a non-negative primitive matrix of size n, then A^{n^2-2n+2} is positive. Furthermore, there exists a primitive matrix B of size n such that B^{n^2-2n+1} is not positive. Equivalently, $\mathrm{p}(n) = n^2 - 2n + 2$.

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Tightness of Wielandt bound

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

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Three primitivities of matrix tuples

Primitivity of matrices plays an important role in the study of markov processes. When we study on some processes which involv multiple matrices (e.g., inhomogeneous Markov process, multi-dimensional Markov process), we need to generalize the concept "primitivity".

There are several possibilities to generalize the concept "primitivity" from a nonnegative matrix to a tuple of nonnegative matrices.

Today, we focus on three generalizations:

- strong primitivity
- primitivity
- Hurwitz primitivity

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Problems

- For a matrix tuple, how to determine whether it is (strongly, Hurwitz) primitive or not?
- For a (Hurwitz) primitive matrix tuple, how to find a positive (Hurwitz) product of it?
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(Strongly) primitive matrix tuples

Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an m-tuple of nonnegative n-by-n matrices. For each finite sequence $\alpha = \alpha_1 \cdots \alpha_k$ over $[m] = \{1, 2 \ldots, m\}$, write $\mathcal{A}_\alpha$ for $A_{\alpha_1} \cdots A_{\alpha_k}$ and call it a product over $\mathcal{A}$ of length k.

- The m-tuple $\mathcal{A}$ is called **primitive** if there exists a finite sequence α over $[m]$ such that

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Types of sequences

Let $\alpha = \alpha_1 \cdots \alpha_k$ be a sequence over a set X.

- For any $x \in X$, we denote the number of **occurrences** of x in the word α by $|\alpha|_x$, that is
$$|\alpha|_x = |\{i \in [k] : \alpha_i = x\}|.$$
- The **type** of α, denoted by $\mathrm{t}(\alpha)$, is the vector in $\mathbb{N}^X$ such that
$$\mathrm{t}(\alpha)(x) = |\alpha|_x$$
for each $x \in X$.

Example

The type of the sequence $\alpha = 1442112$ over $\{1, 2, 3, 4\}$ is
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Hurwitz products and Hurwitz primitivity

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$$\mathcal{A}^\tau = \sum_{\alpha:\ \mathrm{t}(\alpha)=\tau} \mathcal{A}_\alpha \,.$$

We call $\mathcal{A}^\tau$ a **Hurwitz product** of $\mathcal{A}$ of length $|\tau| := \sum_{i=1}^m \tau_i$.

- The tuple $\mathcal{A}$ is **Hurwitz primitive** if it has a positive Hurwitz product.
- The minimum length of positive Hurwitz products is called the **Hurwitz primitive index** of $\mathcal{A}$.

Example

- $\mathcal{A} = (A_1, A_2, A_3)$.
- $\mathcal{A}^{(1,3,0)} = A_1A_2^3 + A_2A_1A_2^2 + A_2^2A_1A_2 + A_2^3A_1$.

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Hierarchy of primitivities

$$\begin{aligned}\{\text{strongly primitive matrix tuple}\} &\subsetneq \{\text{primitive matrix tuple}\}\\ &\subsetneq \{\text{Hurwitz primitive matrix tuple}\}\end{aligned}$$

Determine Problems

- [Gerencsér-Gusev-Jungers[2], 2018] The determine problem of primitivity is NP-hard (even for two matrices).
- The algorithmic complexity of determining Hurwitz primitivity is still unknown.

[2] Balázs Gerencsér, Vladimir V. Gusev и Raphaël M. Jungers (2018). "Primitive sets of nonnegative matrices and synchronizing automata". В: *SIAM J. Matrix Anal. Appl.* 39.1, с. 83—98.

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Maximum (Hurwitz) primitive index

- $\mathrm{p}(n, m) \doteq$ the maximum primitive index of all primitive m-tuples of n-by-n matrices.
- $\mathrm{p}(n) \doteq \max_{m \geq 1} \mathrm{p}(n, m)$.
- $\mathrm{hp}(n, m) \doteq$ the maximum Hurwitz primitive index of all Hurwitz primitive m-tuples of n-by-n matrices.

Theorem (Gerencsér-Gusev-Jungers[3], 2018)

$\lim_{n \to +\infty} \frac{\log \mathrm{p}(n)}{n} = \frac{\log 3}{3}$.

Theorem (Olesky-Shader-Driessche[4], 2002)

$\mathrm{hp}(n, m) = \Theta(n^{m+1})$.

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- $\mathrm{p}(n, m) \doteq$ the maximum primitive index of all primitive m-tuples of n-by-n matrices.
- $\mathrm{p}(n) \doteq \max_{m \geq 1} \mathrm{p}(n, m)$.
- $\mathrm{hp}(n, m) \doteq$ the maximum Hurwitz primitive index of all Hurwitz primitive m-tuples of n-by-n matrices.

Theorem (Gerencsér-Gusev-Jungers[3], 2018)

$\lim_{n \to +\infty} \frac{\log \mathrm{p}(n)}{n} = \frac{\log 3}{3}$.

Theorem (Olesky-Shader-Driessche[4], 2002)

$\mathrm{hp}(n, m) = \Theta(n^{m+1})$.

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Two subfamilies of square matrices

In many applications, the matrices that appear are (doubly) stochastic matrices.

- The set of nonnegative n-by-n matrices that has no zero rows is denoted by $\mathrm{NZ}_1(n)$. (row-stochastic matrix)
- The set of nonnegative n-by-n matrices that has no zero rows and no zero columns is denoted by $\mathrm{NZ}_2(n)$. (doubly-stochastic matrix)

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Block permutation matrices

Let A be an n-by-n matrix. Let $\pi = (\pi_1, \ldots, \pi_r)$ be a partition of $[n]$. We say that A **preserves the partition** π if there exists a permutation $\sigma \in \mathrm{Sym}_r$ such that $A(\pi_i, \pi_j) = 0$ whenever $j \neq \sigma(i)$.

Two characterization theorems

- A tuple of nonnegative matrices $\mathcal{A}$ is **irreducible** if $\sum_{A\in\mathcal{A}} A$ is irreducible.
- A partition is **non-trivial** if it contains at least two parts.

Theorem (Protasov-Voynov[5], 2012)

Let $\mathcal{A}$ be an irreducible tuple of NZ_2-matrices. The tuple $\mathcal{A}$ is not primitive if and only if there exists a non-trivial partition π such that every matrix in $\mathcal{A}$ preserves π.

Theorem (Protasov[6], 2013)

Let $\mathcal{A}$ be an irreducible tuple of NZ_1-matrices. The tuple $\mathcal{A}$ is not Hurwitz primitive if and only if there exists a non-trivial partition π such that every matrix in $\mathcal{A}$ preserves π and all these permutations corresponding to members of $\mathcal{A}$ commute with each other.

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Different proofs

Characterization theorem of primitive $NZ_2(n)$-matrix tuples:

- Protasov-Voynov (2012) give the first proof by using geometrical properties of affine operators on polyhedra.
- Three combinatorial proofs are found by Al'pin-Alpina (2013), Blondel-Jungers-Olshevsky (2015), and Al'pin-Alpina (2019).
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A sketch of the proof (primitive)

Let $\mathcal{A}$ be a m-tuple of nonnegative n-by-n NZ_2-matrices.
Define $\approx$ to be the binary relation on $[n]$ such that $i \approx j$ if for all $i', j' \in [n]$ and for all finite sequence α over $[m]$ satisfying

$$\mathcal{A}_\alpha(i, i') > 0 \quad \text{and} \quad \mathcal{A}_\alpha(j, j') > 0,$$

there exists $k \in [n]$ and a sequence β such that

$$\mathcal{A}_\beta(i', k) > 0 \quad \text{and} \quad \mathcal{A}_\beta(j', k) > 0.$$

The relation $\approx$ is called the **stable relation** of $\mathcal{A}$.
It is routine to verify the following statements.

- The relation $\approx$ is an equivalence relation.
- Let π be the partition which is formed by the equivalence class of $\approx$. The matrices in $\mathcal{A}$ preserve π.
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Let $\mathcal{A}$ be a m-tuple of nonnegative n-by-n NZ_1-matrices.

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Algorithms

We can determine a given NZ_1-matrix tuple (resp., NZ_2-matrix tuple) whether is primitive (resp., Hurwitz primitive) or not by calculating the equivalence relation $\approx$ (resp., $\overset{h}{\approx}$).

Then there is an algorithm

- to determine primitivity for a given NZ_1-matrix tuple in $O(n^2 m)$-time;
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Maximum (Hurwitz) primitive index

Let X be a subfamily of nonnegative matrices.

- $\mathrm{p}_X(n) \doteq$ the maximum primitive index of all primitive tuples of n-by-n X-matrices;
- $\mathrm{hp}_X(n) \doteq$ the maximum Hurwitz primitive index of all Hurwitz primitive tuples of n-by-n X-matrices.

We will present some results on $\mathrm{p}_{\mathrm{NZ}_2}(n)$ and $\mathrm{hp}_{\mathrm{NZ}_1}(n)$.

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$\mathrm{p}_{\mathrm{NZ}_2}(n)$ and $\mathrm{hp}_{\mathrm{NZ}_1}(n)$

- [Blondel-Jungers-Olshevsky[7], 2015]

$$\frac{n^2}{2} \le \mathrm{p}_{\mathrm{NZ}_2}(n) \le 2\mathrm{c}(n) + n - 1 \le O(n^3).$$

- [Gusev[8], 2013]

$$(n-1)^2 \le \mathrm{hp}_{\mathrm{NZ}_1}(n).$$

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$p_{NZ_2}(n)$ and $hp_{NZ_1}(n)$

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Synchronizing automata

- A square Boolean matrix is called an **automaton matrix** if each row of A contains a unique 1.
- An n-state **automaton** is a tuple of n-by-n automaton matrices.
- An automaton $\mathcal{A}$ is **synchronizing** if there exists a product $\mathcal{A}_\alpha$ which contains a positive column.
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Example (4-state synchronizing automaton)

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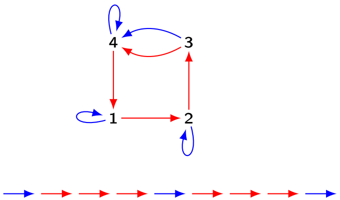

Černý Conjecture

Define the **Černý function** $c(n)$ as the maximum synchronizing index of all synchronizing automata with n states.

Conjecture (Černý, 1971[10])

$c(n) = (n-1)^2$.

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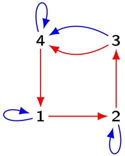

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There are some upper bounds of $c(n)$ which roughly equals $\frac{n^3}{6}$.

- [Frankl[12]-Pin[13] 1982] $c(n) \leq \frac{n^3-n}{6} \approx 0.167n^3$
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Some progresses on Černý Conjecture, Cont'd

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Connection between primitive NZ_2-matrix tuples and Synchronizing Automata

Let $\mathcal{A}$ be an primitive tuple of n-by-n Boolean NZ_2-matrix.

- $\mathcal{C} \doteq \{C : C \leq A \in \mathcal{A} \text{ and } C \text{ is an automaton matrix}\}$.

Observation (Blondel-Jungers-Olshevsky[16], 2015)

The automaton $\mathcal{C}$ is synchronizing.

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Proof of the upper bound of $\mathrm{hp}_{\mathrm{NZ}_1}(n)$

- Regard $\mathcal{A} = (A_1, \ldots, A_m)$ as an arc-labeled digraph D, where $V(D) = [n]$ and $E(D) = \{x \xrightarrow{k} y : A_k(x, y) > 0\}$.
- Find a positive Hurwitz product of $\mathcal{A}$ $\Leftrightarrow$ find $\tau \in \mathbb{N}^m$ such that for all vertices x and y there exists a walk from x to y such that the arc-label sequence of this walk is type-τ.
- By the observation in the last page, there exists $\tau' \in \mathbb{N}^m$ and a vertex z such that for each vertex x there exists a walk from x to z satisfying the arc-label sequence of this walk is type-τ' and $|\tau'| \leq 2\,\mathrm{c}(n)$.
- Since the digraph D is strongly connected, there exists a closed walk W which visits every vertex and has length at most $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$.
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Strongly primitive matrix tuples

Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an m-tuple of nonnegative n-by-n matrices.

- The m-tuple $\mathcal{A}$ is called **strongly primitive** if there exists a positive integer k such that for all length-k sequence α over $[m]$ such that

$$\mathcal{A}_\alpha > 0.$$

The minimum such integer k is called the **strongly primitive index** of $\mathcal{A}$, denoted by $\mathrm{sp}(\mathcal{A})$.

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Maximum strongly primitive matrix index

Theorem (Cohen-Sellers[17], 1982)

For any strongly primitive n-by-n matrix tuple $\mathcal{A}$,

$$\mathrm{sp}(\mathcal{A}) \leq 2^n - 2.$$

Moreover, there exists a strongly primitive n-by-n matrix tuple $\mathcal{B}$ such that $\mathrm{sp}(\mathcal{B}) = 2^n - 2$.

- Define $\gamma(n) := \min\{|\mathcal{A}| : g(\mathcal{A}) = 2^n - 2, \mathcal{A} \text{ is an order-}n \text{ strongly primitve matrix set}\}$.
- The Cohen-Sellers construction shows $\gamma(n) \leq 2^n - 3$.
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Our construction

Define $\mathcal{A}_n = \{A_1, \ldots, A_n\}$ such that

$$A_k(i,j) = \begin{cases} 1 & \text{if either } i = j \text{ or } i = k \text{ or } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

One can check that $\mathrm{sp}(\mathcal{A}_n) = 2^n - 2$.

Example

$$\mathcal{A}_4 = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right).$$

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Motivation

- It is not yet known whether there is a polynomial-time algorithm for determining strong primitivity.
- Cohen-Seller's construction tells us that the strong primitivity index $\mathrm{sp}(n)$ can grow exponentially.
- To understand the strong primitivity, an important question is whether the strong primitivity index $\mathrm{sp}(n, m)$ can grow exponentially with respect to n and m.
- Our construction shows that $\mathrm{sp}(n, n)$ grows exponentially. This may be a sign that the strong primitivity can hardly be determined in polynomial time.

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Summary

	Primitivity		Hurwitz Primitivity		Strong primitivity
Assumption		NZ_2		NZ_1	
Determine problem	NP-hard	$O(n^2 m)$	?	$O(n^2 m^2 + n^3 m)$	?
Finding such a product	NP-hard	$O(n^3 m)$	?	$O(n^3 m^2)$	/
Lower bounds of indices	$3^{\frac{n}{3}(1-\epsilon)}$	$\frac{n^2}{2}$	Cn^{m+1}	$(n-1)^2 + 1$	$2^n - 2$
Upper bounds of indices	$3^{\frac{n}{3}(1+\epsilon)}$	$O(n^3)$	$m!mn^{m+1} + n^2$	$O(n^3)$	$2^n - 2$

Thank you

Спасибо