Automata with an almost constant rank property on words

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Deterministic automata

A **(deterministic) automaton** $\mathcal{A}$ is a 3-tuple (Q, Σ, δ), consisting of

- a set of states Q;
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Let Σ^* be the set of all finite words (sequences) over Σ. Use $\varepsilon \in \Sigma^*$ to denote the word of length zero. The transition function naturally extends to a function $Q \times \Sigma^* \to Q$ (also denoted by δ) such that for all $\alpha = (\alpha_1, \ldots, \alpha_t) \in \Sigma^*$ and $q \in Q$,

$$\delta(q, \alpha) = \begin{cases} q, & \text{if } \alpha = \varepsilon; \\ \delta(q, \alpha), & \text{if } t = 1; \\ \delta(\delta(q, (\alpha_1, \ldots, \alpha_{t-1})), \alpha_t), & \text{if } t > 1. \end{cases}$$

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To simplify notation we often write $q.\alpha$ for $\delta(q, \alpha)$ and $P.\alpha$ for $\{\delta(q, \alpha) : q \in P\}$, where $q \in Q$, $P \subseteq Q$ and $\alpha \in \Sigma^*$.

Automata and arc-colored digraphs

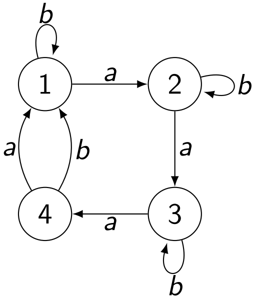
We illustrate an automaton as an arc-colored digraph with one arc of each color out of each vertex. Vertices are states, colors are transitions.

Example

Let $\mathcal{A}$ be the automaton $(\{1,2,3,4\},\{a,b\},\delta)$ such that

x	1	2	3	4
$x.a$	2	3	4	1
$x.b$	1	2	3	1

$1.aaab = 1 = 2.aaab$



Synchronizing words

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. A word $\alpha \in \Sigma^*$ is **synchronizing** if it sends any state of $\mathcal{A}$ to the same state, that is $q.\alpha = q'.\alpha$ for all $q, q' \in Q$. The minimum length of synchronizing words for $\mathcal{A}$ is called the **synchronizing threshold** of $\mathcal{A}$.
An automaton is **synchronizing** if it has a synchronzing word.

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In 1964, Černý starts to study the synchronzing threshold of automata.

Define the **Černý function** $C(n)$ as the maximum synchronizing threshold of all synchronizing automata with n states.

Conjecture (Černý, 1971[1])

$C(n) = (n-1)^2$.

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Some progresses on Černý Conjecture

In 1964, Černý[2] found a family of automata $\{\mathcal{C}_n\}$ such that $\mathcal{C}_n$ is an n-state synchronizing automaton whose synchronizing threshold equals $(n-1)^2$. This shows that $(n-1)^2 \leq C(n)$.

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[3] P. Frankl (1982). "An extremal problem for two families of sets". In: *European J. Combin.* 3.2, pp. 125–127. ISSN: 0195-6698. DOI: 10.1016/S0195-6698(82)80025-5.

[4] J.-E. Pin (1983). "On two combinatorial problems arising from automata theory". In: *Combinatorial mathematics (Marseille-Luminy, 1981).* Vol. 75. North-Holland Math. Stud. North-Holland, Amsterdam, pp. 535–548.

[5] Marek Szykuła (2018). "Improving the upper bound and the length of the shortest reset words". In: *35th Symposium on Theoretical Aspects of Computer Science.* Vol. 96. LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, Art. No. 56, 13. DOI: 10.4230/LIPIcs.STACS.2018.56.

[6] Yaroslav Shitov (2019). "An Improvement to a Recent Upper Bound for Synchronizing Words of Finite Automata". In: *Journal of Automata, Languages and Combinatorics* 24.2–4, pp. 367–373. DOI: 10.25596/jalc-2019-367.

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There are some upper bounds of $C(n)$ which approximately equals $O(\frac{n^3}{6})$.

- [Frankl[3]-Pin[4] 1982] $C(n) \leq \frac{n^3-n}{6} \leq O(0.16667n^3)$
- [Szykuła[5] 2018] $C(n) \leq \frac{85059n^3+90024n^2+196504n-10648}{511104} \leq O(0.16643n^3)$
- [Shitov[6] 2019] $C(n) \leq \left(\frac{7}{48} + \frac{15625}{798768}\right) n^3 + o(n^2) \leq O(0.16540n^3)$

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Eventual ranges and rank

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. A subset $P \subseteq Q$ is an **eventual range** of $\mathcal{A}$ if

- there exists a word $\alpha \in \Sigma^*$ such that $P = Q.\alpha$;
- for all $p, q \in P$ and $\beta \in \Sigma^*$, if $p \neq q$ then $p.\beta \neq q.\beta$.

Use $\mathrm{ev}(\mathcal{A})$ to denote the set of eventual range of $\mathcal{A}$.

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Use $\mathrm{ev}(\mathcal{A})$ to denote the set of eventual range of $\mathcal{A}$. An **eventual-range word** of $\mathcal{A}$ is a word $\alpha \in \Sigma^*$ such that $Q.\alpha \in \mathrm{ev}(\mathcal{A})$. The minimum length of eventual-range words for $\mathcal{A}$ is called the **eventual-range threshold** of $\mathcal{A}$.

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Assume that the state set of $\mathcal{A}$ is finite. Observe that the size of eventual ranges of $\mathcal{A}$ are equal. This size is called the **rank** of $\mathcal{A}$. Define the function $C(n, r)$ as the maximum eventual-range threshold of all n-state automata of rank r.

Volkov's Deficiency Conjecture

Conjecture (Volkov[7] 2004)

For $n \geq r$, $C(n, r) = (n - r)^2$.

Remark

Since an automaton is synchronizing if and only if its rank equals 1, the above conjecture generalizes the Černý Conjecture.

[7] S. W. Margolis, J.-E. Pin, and M. V. Volkov (2004). "Words guaranteeing minimum image". In: *Internat. J. Found. Comput. Sci.* 15.2, pp. 259–276. ISSN: 0129-0541. DOI: 10.1142/S0129054104002406.

An analog problem of Volkov's Deficiency Conjecture

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. We say that $\mathcal{A}$ has **Property S** if there exists an integer k such that every word over Σ of length $\geq k$ is an **eventual-range word** of $\mathcal{A}$.
Use $\mathbb{S}$ to denote the family of automata with Property S.
If $\mathcal{A} \in \mathbb{S}$, the **strongly eventual-range threshold** of $\mathcal{A}$, denoted by $\mathrm{sevt}(\mathcal{A})$, is defined as the minimum integer k such that $Q.\alpha \in \mathrm{ev}(\mathcal{A})$ for every word $\alpha \in \Sigma^{\geq k}$.

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Define the function $D(n, r)$ to be the maximum strongly eventual-range threshold of all n-state automata of rank r in $\mathbb{S}$.

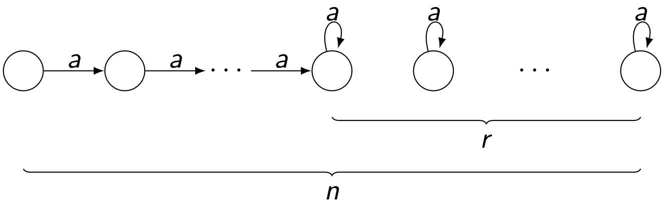
Conjecture (Wu-Z., 2022+)

For $n \geq r$, $D(n, r) = n - r$.

A lower bound of $D(n, r)$

The following example shows that $D(n, r) \geq n - r$.

Example



A classification of $\mathbb{S}$

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. Let $\mathcal{A}^{(2)}$ be the automaton (Q', Σ, δ) such that

- $Q' = \binom{Q}{1} \cup \binom{Q}{2}$;
- $\delta(P, a) = P.a = \{\delta(p, a) : p \in P\}$ for all $P \in Q'$ and $a \in \Sigma$.

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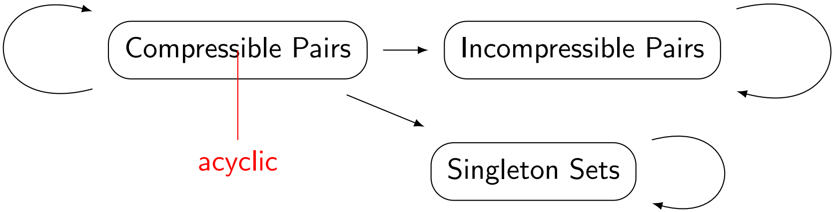
A pair of states p, q in $\mathcal{A}$ is **compressible** if there exists $\alpha \in \Sigma^*$ such that $p.\alpha = q.\alpha$. Use $\mathrm{com}(\mathcal{A})$ to denote the set of compressible pairs.

Theorem (Wu-Z., 2022+)

Let G be the arc-colored digraph correponding to $\mathcal{A}^{(2)}$. Then $\mathcal{A} \in \mathbb{S}$ if and only if the induced subdigraph $G[\mathrm{com}(\mathcal{A})]$ of G on $\mathrm{com}(\mathcal{A})$ is acyclic.



Some remarks on the classification

Let $\mathcal{A}$ be an automaton in $\mathbb{S}$.

- The acyclic property allows us to define a partial order $\preceq_{\mathcal{A}}$ on $\mathrm{com}(\mathcal{A})$ as $P_1 \preceq P_2$ if there exists $\alpha \in \Sigma^*$ such that $P_1 = P_2.\alpha$.

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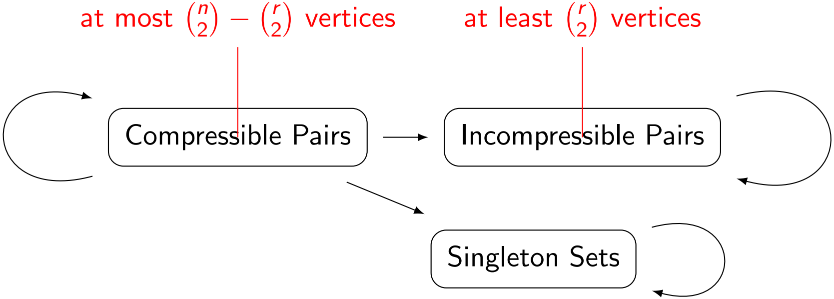
- The maximum size of a chain (a totally ordered subset) in the poset $(\mathrm{com}(\mathcal{A}), \preceq_{\mathcal{A}})$ equals the strongly eventual-range threshold of $\mathcal{A}$.
- For $n \geq r$, $D(n, r) \leq \binom{n}{2} - \binom{r}{2}$.



Congruences and quotient automata

An equivalence relation ρ on the state set Q of an automata $\mathcal{A} = (Q, \Sigma, \delta)$ is called a **congruence** if $(p, q) \in \rho$ implies $(p.a, q.a) \in \rho$ for all $p, q \in Q$ and $a \in \Sigma$.

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For a congruence ρ of $\mathcal{A}$, the **quotient automaton** $\mathcal{A}/\rho$ is the automaton $(Q', \Sigma, \delta_\rho)$ where

- $Q' = \{[q]_\rho : q \in Q\}$;
- $\delta_\rho([q]_\rho, a) = [\delta(q, a)]_\rho$.

Rank 1

Theorem (Bleak-Cameron-Maissel-Navas-Olukoya[8] 2019)

For $n \geq 1$, $D(n, 1) = n - 1$.

Proof.

We apply induction on n. For $n = 1$, it clearly holds $D(1, 1) = 0$.

[8] Collin Bleak et al. (2019). *The further chameleon groups of Richard Thompson and Graham Higman: Automorphisms via dynamics for the Higman groups $G_{n,r}$.* arXiv: 1605.09302 [math.GR].

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- $\mathcal{A}/\rho \in \mathbb{S}$;
- $\mathrm{rank}(\mathcal{A}/\rho) = 1$;
- $\mathrm{sevt}(\mathcal{A}) \leq \mathrm{sevt}(\mathcal{A}/\rho) + 1$.

By induction assumption, $\mathrm{sevt}(\mathcal{A}) \leq \mathrm{sevt}(\mathcal{A}/\rho) + 1 \leq n - 2 + 1 = n - 1$. Hence $D(n, 1) = n - 1$. □

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Eventual range automata

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton. Let $\mathcal{EV}(\mathcal{A}) = (\mathrm{ev}(\mathcal{A}), \Sigma, \delta)$ be the **eventual range automaton** of $\mathcal{A}$ such that

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Lemma

Let $\mathcal{A}$ be an automaton in $\mathbb{S}$. Then $\mathcal{EV}(\mathcal{A})$ is an automaton of rank 1 in $\mathbb{S}$. Moreover, if $\mathcal{A}$ is irreducible, then $\mathrm{sevt}(\mathcal{A}) = \mathrm{sevt}(\mathcal{EV}(\mathcal{A}))$.

Rank 2

Theorem (Wu-Z., 2022+)

For $n \geq 2$, $D(n, 2) = n - 2$.

Proof.

We apply induction on n. For $n = 2$, it clearly holds $D(2, 2) = 0$.

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CASE 1. All eventual ranges of $\mathcal{A}$ are pairwise disjoint.

We have $\mathrm{sevt}(\mathcal{A}) = \mathrm{sevt}(\mathcal{EV}(\mathcal{A})) \leq |\,\mathrm{ev}(\mathcal{A})| - 1 \leq \frac{n}{2} - 1 \leq n - 2$.

Rank 2, Cont'd

CASE 2. There exist $P_1, P_2 \in \mathrm{ev}(\mathcal{A})$ such that $|P_1 \cap P_2| = 1$.

Take a minimal element $\{T_1, T_2\}$ in the poset $(\mathrm{com}(\mathcal{EV}(\mathcal{A})), \preceq_{\mathcal{EV}(\mathcal{A})})$ such that $\{T_1, T_2\} \preceq_{\mathcal{EV}(\mathcal{A})} \{P_1, P_2\}$.

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Rank 2, Cont'd

CASE 2. There exist $P_1, P_2 \in \mathrm{ev}(\mathcal{A})$ such that $|P_1 \cap P_2| = 1$.

Take a minimal element $\{T_1, T_2\}$ in the poset $(\mathrm{com}(\mathcal{EV}(\mathcal{A})), \preceq_{\mathcal{EV}(\mathcal{A})})$ such that $\{T_1, T_2\} \preceq_{\mathcal{EV}(\mathcal{A})} \{P_1, P_2\}$. Observe that $|T_1 \cap T_2| = 1$. Assume that $T_1 = \{p, x\}$ and $T_2 = \{q, x\}$ satisfying $p \neq q$. Since $\mathrm{rank}(\mathcal{EV}(\mathcal{A})) = 1$, it holds that $T_1.a = T_2.a$ for all $a \in \Sigma$. Therefore $p.a = q.a$ for all $a \in \Sigma$.
Define $\rho = \{(x, x) : x \in Q\} \cup \{(p, q), (q, p)\}$. Observe that ρ is a congruence of $\mathcal{A}$.
One can check that

- $\mathcal{A}/\rho \in \mathbb{S}$;
- $\mathcal{A}/\rho$ is irreducible;
- $\mathrm{rank}(\mathcal{A}/\rho) = 2$;
- $\mathrm{sevt}(\mathcal{A}) \leq \mathrm{sevt}(\mathcal{A}/\rho) + 1$.

By induction assumption, $\mathrm{sevt}(\mathcal{A}) \leq \mathrm{sevt}(\mathcal{A}/\rho) + 1 \leq n - 3 + 1 = n - 2$. $\square$

Rank 3

Theorem (Wu-Z., 2022+)

For $n \geq 3$, $D(n,3) = o(n^2)$.

Proof.

We apply induction on n. For $n = 3$, it clearly holds $D(3,3) = 0$.
Let $\mathcal{A} = (Q, \Sigma, \delta) \in \mathbb{S}$ be an n-state irreducible automaton of rank 3. (the proof of the non-irreducible case is similar, but involves more technical details.)

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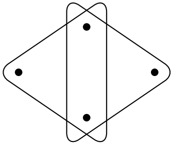
CASE 1. There exists $P_1, P_2, P_3 \in \mathrm{ev}(\mathcal{A})$ such that one of the following conditions holds:

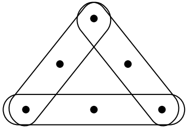
- $|P_1 \cap P_2| = 2$;
- $|P_1 \cap P_2| = |P_2 \cap P_3| = |P_3 \cap P_1| = 1$ and $P_1 \cap P_2 \cap P_3 = \emptyset$.

By using some combinatorial arguments, we can show that there exists a pair of states $p, q \in Q$ such that $p.a = q.a$ for all $a \in \Sigma$. Using the similar arguments in the previous proofs, we can obtain that $\mathrm{sevt}(\mathcal{A}) \leq o(n^2)$.

Triangle Removal Lemma

Define $f(n)$ as the maximum number of hyperedges in a 3-uniform hypergraph which does not have a sub-hypergraph isomorphic to F_1 or F_2.

F_1 : 

F_2 : 

Theorem

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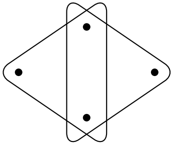
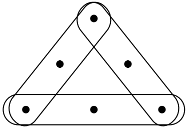


F_2 :



Theorem

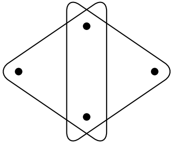
$f(n) = o(n^2)$.

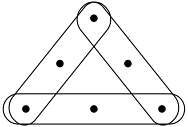
This theorem is a corollary of the Triangle Removal Lemma.

Theorem (Triangle Removal Lemma)

Every graph on n vertices with $o(n^3)$ triangles (the complete graph on three vertices) can be made triangle-free by removing at most $o(n^2)$ edges.

Rank 3, Cont'd

F_1 : 

F_2 : 

CASE 2. For all distinct $P_1, P_2, P_3 \in \mathrm{ev}(\mathcal{A})$ such that none of the following conditions holds

- $|P_1 \cap P_2| = 2$;
- $|P_1 \cap P_2| = |P_2 \cap P_3| = |P_3 \cap P_1| = 1$ and $P_1 \cap P_2 \cap P_3 = \emptyset$.

Consider the 3-uniform hypergraph H whose vertex set is Q and hyperedge set is $\mathrm{ev}(\mathcal{A})$. Clearly, H has no subgraph isomorphic to F_1 or F_2. Then $\mathrm{sevt}(\mathcal{A}) \leq \mathrm{sevt}(\mathcal{EV}(\mathcal{A})) \leq |\mathrm{ev}(\mathcal{A})| - 1 = (\text{the number of hyperedges in } H) - 1 = o(n^2)$. $\square$

$D(n, r)$

We conclude our knowledge of $D(n, r)$ as the following: for $n \geq r$,

$$D(n, r) \begin{cases} = n - r & \text{if } r \in \{1, 2, n-2, n-1, n\}; \\ = o(n^2) & \text{if } r = 3; \\ \leq \binom{n}{2} - \binom{r}{2} & \text{if } 4 \leq r \leq n-3. \end{cases}$$

A generalization

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be an automaton such that Q is an n-dimensional vector space over a field $\mathbb{F}$ and $\delta(\cdot, a)$ is an affine linear transformation on Q for all $a \in \Sigma$. Observe that every eventual range of $\mathcal{A}$ is an affine subspace of Q and they have the same dimension, denoted by r.

Theorem (Wu-Z. 2022+)

If $\mathcal{A} \in \mathbb{S}$ and $r \in \{0, 1\}$, then $\mathrm{sevt}(\mathcal{A}) \leq n - r$.

Remark

The above theorem is a generalization of the results $D(n, 1) = n - 1$ and $D(n, 2) = n - 2$.

Thank you!