Path-liftable digraph homomorphisms and non-liftable indices

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A digraph is a quadruple \((V, E, i, t)\):

- vertex set \(V\); arc set \(E\);
- initial operator \(i : E \rightarrow V\); terminal operator \(t : E \rightarrow V\).

A digraph homomorphism from a digraph \(G\) to a digraph \(H\) is a pair of maps \(\phi = (\phi_0, \phi_1)\) such that the following diagrams commute.
Liftings

A digraph homomorphism $\phi \in \text{hom}(G, H)$ is $K$-liftable if for every $\alpha \in \text{hom}(K, H)$ there exists $\beta \in \text{hom}(K, G)$ such that $\alpha = \phi \circ \beta$. 

\[
\begin{array}{c}
G \\
\downarrow_{\phi} \quad \exists \beta \\
H \leftarrow \downarrow_{\alpha} \quad K
\end{array}
\]
Path-liftable property

$P_k$ denotes the directed path digraph of length $k$.

\[ \cdots \Rightarrow P_2 \text{-liftable} \Rightarrow P_1 \text{-liftable}. \]

A digraph homomorphism is **path-liftable** if it is $P_k$-liftable for every $k$. A homomorphism in $\text{hom}(P_k, G)$ is called a $k$-**walk** of $G$.

If $\phi$ is not path-liftable, the **non-liftable index** of $\phi$ is

\[ \delta(\phi) := \text{the length of shortest walk in } H \text{ without any lifting} \]
An example

A non-liftable 8-walk
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Question

For given strongly connected digraphs $G, H$ and $\phi : G \rightarrow H$, how to determine whether $\phi$ is path-liftable or not?

Question

Can we bound $\delta(\phi)$ by the size of $G$ and $H$?

We will consider the two questions in the following case:

- general case;
- isentropic case, $\lambda_G = \lambda_H$;
- $G$ and $H$ are De Bruijn / Kautz digraphs.
A upper bound of non-liftable indices

For $\phi : G \to H$,

$$\delta(\phi) \leq 2|V_G| - 1.$$ 

**Proof.**

- Pick a shortest non-liftable walk $(e_1, e_2, \ldots, e_k)$ in $H$.
- $R_0 := \phi_0^{-1}(i(e_1))$
- $R_i := \{\text{the terminal vertices of liftings of } (e_1, \ldots, e_i)\}$
- If $R_i = R_j$ and $i < j$, then the walk $(e_1, \ldots, e_i, e_{j+1} \ldots, e_k)$ is also a non-liftable walk.
- $R_0, R_1 \ldots, R_k$ are distinct subsets of $V_G$. Thus $\delta(\phi) = k \leq 2|V_G| - 1$.

□

This bound is tight.
Let $\mathbb{B}$ be the Boolean semiring. Let $\prec$ be the reverse lexicographical order on $\mathbb{B}^n$ which is define by $x \prec y$ if $x(i) > y(i)$ for the minimum $i$ where $x(i) \neq y(i)$.

**Example**

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
For $k \in [n]$, define

$$A_k = \begin{pmatrix}
[k-1] & \{k\} & [k+1, n] \\
\{k\} & I & 0 & 0 \\
[k+1, n] & 0 & 0 & J \\
J & J & J & J
\end{pmatrix}.$$ 

Let $\mathbb{B}^n = \{\pi_i : i \in [2^n]\}$ such that $\pi_i \preceq \pi_j$ if $i \preceq j$. One can check that

$$A_k \pi_i \begin{cases} 
\preceq \pi_i & \text{if } k \neq p, \\
= \pi_{i+1} & \text{if } k = p.
\end{cases}$$

Then $0 \in \langle A_k : k \in [n] \rangle$ and any product of $2^n - 2$ elements in \{A_k : k \in [n]\} is not 0.
Construct $\phi : G \rightarrow H$ as follow.

- Let $H$ be the digraph such that $V_H = \{1\}$ and $E_H = [n]$.
- Let $G$ be the digraph such that
  - $V_G = [n]$;
  - $E_G = \{(i, j, k) : A_k(i, j) = 1\}$;
  - initial operator is defined by $i_G((i, j, k)) = i$;
  - terminal operator is defined by $t_G((i, j, k)) = j$.
- Let $\phi : G \rightarrow H$ be the homomorphism such that $\phi_1((i, j, k)) = k$.
- For a walk $(e_1, \ldots, e_k)$ in $H$,
  \[
  \# \text{ liftings of } (e_1, \ldots, e_k) = \# 1 \text{ in } A_{e_1} A_{e_2} \cdots A_{e_k}.
  \]
- Then $\delta(\phi) = 2^n - 1 = 2^{|V_G|} - 1$. 
A dichotomy

Let $H$ be a fixed strongly connected digraph.

**Question**

*Input a digraph $G$ and $\phi : G \rightarrow H$. What is the complexity to determine whether $\phi$ is path-liftable or not?*

If $H$ is a cycle, it is easy ($G$ has non-trivial strongly connected components $\iff \phi$ is path-liftable).

**Theorem (Wu, Z.)**

*If $H$ is not a cycle, then the determine problem is NP-complete.*

- A reduction from 3-SAT problem. **blackboard**
Three operators on Boolean semi-field.

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Table: Negation

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Table: Conjunction

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Table: Disjunction

Let \(x_1, \ldots, x_n\) be Boolean variables. A literal is either a variable or the negation of a variable. A clause is the disjunction of three literals. A 3-CNF formula is the conjunction of clauses.

Example

\[(x_1 \lor x_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_2 \lor x_4)\]
3-SAT problem

For given a 3-CNF formula $\mathcal{F}$, the 3-SAT problem is whether or not an assignment of the variables that make $\mathcal{F} = 1$.

**Theorem (Cook, 1971)**

3-SAT problem is NP-complete.
Spectral radius

Let $G$ and $H$ be digraphs.

- $\lambda_G$: the spectral radius of the adjacency matrix of $G$.
- Note that

$$\lambda_G = \lim_{k \to +\infty} \frac{1}{k} \log \left( |\text{hom}(P_k, G)| \right).$$

- Thus, if $\lambda_G < \lambda_H$, there is no path-liftable homomorphism from $G$ to $H$.
- What is the phenomenon when $\lambda_G = \lambda_H$?
Let \( \phi \in \text{hom}(G, H) \) and \( \gamma, \gamma' \in \text{hom}(P_k, G) \). We call \( (\gamma, \gamma') \) a **diamond** of \( \phi \) if

- distinct: \( \gamma \neq \gamma' \);
- same image: \( \phi \circ \gamma = \phi \circ \gamma' \);
- same initial vertex: \( i(\gamma) = i(\gamma') \);
- same terminal vertex: \( t(\gamma) = t(\gamma') \).

**Figure:** a diamond
A cubic-time algorithm

Theorem (Well known in symbolic dynamic)

Let $G$ and $H$ be two strongly connected digraphs and $\phi \in \text{hom}(G, H)$, then any two of the following expressions implies the other one.

1. $\lambda_G = \lambda_H$.
2. $\phi$ is path-liftable.
3. $\phi$ has no diamond.

Theorem (Even$^1$, 1965)

There is an algorithm to determine whether a homomorphism $\phi \in \text{hom}(G, H)$ has a diamond or not in time $O(|V_G|^3)$.

De Bruijn and Kautz digraphs

- $K^+_n$: $n$-vertex complete digraph with loops.
- $d$-dimension De Bruijn digraph $B(n, d)$: the $(d - 1)$-th line digraph of $K^+_n$.
- $K_n$: $n$-vertex complete digraph without loops.
- $d$-dimension Kautz digraph $K(n, d)$: the $(d - 1)$-th line digraph of $K_n$. 
Motivation

**Definition (Tvrdik, Harbane and Heydemann\(^2\), 1998)**

Let \( d \) be an integer, \( d \geq 2 \). Let \( \diamond \) be a binary operation on \( \mathbb{Z}_n \) such that for any \( y_1, \ldots, y_{d-1} \in \mathbb{Z}_n \), the set of \( d - 1 \) equations

\[
x_i \diamond x_{i+1} = y_i, \quad 1 \leq i \leq d - 1
\]

for unknowns \( x_1, \ldots, x_d \) has exactly \( n \) distinct solutions such that \( x_i \in \mathbb{Z}_n \). Then it is said that the operation \( \diamond \) satisfies Property \((P_d)\).

Their problem is to find all binary operations on \( \mathbb{Z}_n \) satisfying Property \((P_d)\) for all \( d \).

**Example**

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**Table:** An operation satisfies Property \((P_d)\) for all \(d\).
Let $\diamond$ be a binary operation on $\mathbb{Z}_n$

- $\diamond$ is corresponding to the digraph homomorphism $\phi : B(n, 2) \to B(n, 1)$ such that $\phi_0(a, b) = a \diamond b$.
- One can show that $\diamond$ satisfies Property $(P_d)$ for all $d$ if and only if $\phi$ is path-liftable.
- Tvrdik, Harbane and Heydemann also consider a variant definition which is corresponding to the digraph homomorphism from $K(n, 2)$ to $K(n, 1)$. 
Right-covering (Left-covering) homomorphism

- $\phi \in \text{hom}(G, H)$
- for $v \in V_G$, define $G^+(v) = \{ e \in E_G : i(e) = v \}$.

$\phi$ is called right-covering if $\phi_1$ is surjective from $G^+(v)$ to $H^+(\phi_0(v))$ for all $v \in V_G$.

By symmetry, we define left-covering homomorphisms. We call $\phi$ a one-sided covering if it is either a right-covering or a left-covering or both.

One-sided covering is always path-liftable.
Conjecture (Tvrdik, Harbane, Heydemann, 1998)

Let \( n \) be a prime and let \( \phi \in \text{hom}(B(n, 2), B(n, 1)) \) and \( \psi \in \text{hom}(K(n + 1, 2), K(n + 1, 1)) \).

- If \( \phi \) is path-liftable, then \( \phi \) is one-sided covering.
- If \( \psi \) is path-liftable, then \( \psi \) is one-sided covering.
- If \( \psi \) is not path-liftable, then \( \delta(\psi) \leq 3 \).

Theorem (Wu,Z.)

Let \( G \) and \( H \) be two \( k \)-regular strongly connected digraphs. If \( \frac{|V_G|}{|V_H|} \) is a prime number, then \( \phi \) is path-liftable iff it is a one-sided covering.

- Define three positive integer parameters \( M(\phi), R(\phi) \) and \( L(\phi) \).
- \( L(\phi)M(\phi)R(\phi) = \frac{|V_G|}{|V_H|} \) is a prime.
- Either \( L(\phi) \) or \( R(\phi) \) equals 1. Thus \( \phi \) is a one-sided covering.
Degree

• Let $G, H$ be two strongly connected $k$-regular digraphs.

• A bi-infinite walk $\tau \in \text{hom}(P_\infty, G)$ is **doubly transitive** if for every finite walk $\gamma$ in $G$, it occurs in $\tau$ infinite many times in both directions.

$$\tau = \cdots * * * \gamma * * * \gamma * * * * * * * \gamma * * * * * \cdots$$

• $\phi \in \text{hom}(G, H)$

**Lemma**

There exists a positive integer $M(\phi)$ such that

$$M(\phi) = |\{\alpha \in X_G : \phi \circ \alpha = \tau\}|$$

for all doubly transitive walk $\tau$.

We call the number $M(\phi)$ the **degree** of $\phi$. 
Welch indices

Let $G$ and $H$ be two strongly connected $k$-regular digraphs. Let $\phi \in \text{hom}(G, H)$ be path-liftable homomorphism. For a finite walk $\gamma$ in $G$ and a finite walk $\tau$ in $H$, define the $\phi$-compatible right extension of $(\gamma, \tau)$ to be

$$\mathcal{R}_\phi(\gamma, \tau) \doteq \{ t(\gamma \gamma') : \phi \circ (\gamma \gamma') = \tau \}.$$ 

Define $R_\phi(\gamma) = \max_{\tau} \{|\mathcal{R}_\phi(\gamma, \tau)|\}$ and $L_\phi(\gamma) = \max_{\tau} \{|\mathcal{L}_\phi(\gamma, \tau)|\}$. 
Welch indices, cont’d

- [Hedlund³, 1969] There exists integer $R(\phi)$ and $L(\phi)$ such that $R_\phi(\gamma) = R(\phi)$ and $L_\phi(\gamma) = L(\phi)$ for all $\gamma$.

- [Hedlund, 1969] $L(\phi)M(\phi)R(\phi) = \frac{|V_G|}{|V_H|}$.

- In Hedlund’s paper, $G = B(n, k)$ and $H = B(n, k')$. The proof is also valid for $k$-regular case.