Path-liftable digraph homomorphisms and non-liftable indices

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Digraphs and digraph homomorphisms

A digraph is a quadruple $(\mathrm{V}, \mathrm{E}, \mathrm{i}, \mathrm{t})$:

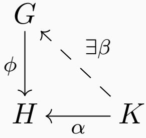
- vertex set V; arc set E;
- initial operator $\mathrm{i} : \mathrm{E} \to \mathrm{V}$; terminal operator $\mathrm{t} : \mathrm{E} \to \mathrm{V}$.

A digraph homomorphism from a digraph G to a digraph H is a pair of maps $\phi = (\phi_0, \phi_1)$ such that the following diagrams commute.

$$\begin{array}{ccc} \mathrm{E}_G & \xrightarrow{\mathrm{i}_G} & \mathrm{V}_G \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_0} \\ \mathrm{E}_H & \xrightarrow{\mathrm{i}_H} & \mathrm{V}_H \end{array} \qquad \begin{array}{ccc} \mathrm{E}_G & \xrightarrow{\mathrm{t}_G} & \mathrm{V}_G \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_0} \\ \mathrm{E}_H & \xrightarrow{\mathrm{t}_H} & \mathrm{V}_H \end{array}$$

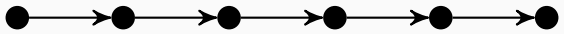
Liftings

A digraph homomorphism $\phi \in \hom(G, H)$ is K-liftable if for every $\alpha \in \hom(K, H)$ there exists $\beta \in \hom(K, G)$ such that $\alpha = \phi \circ \beta$.



Path-liftable property

P_k denotes the directed path digraph of length k.



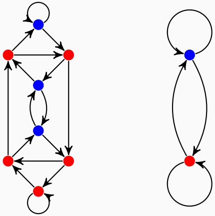
A digraph homomorphism is path-liftable if it is P_k-liftable for every k. A homomorphism in $\hom(\mathrm{P}_k, G)$ is called a k-walk of G.

$$\cdots \Rightarrow \mathrm{P}_2\text{-liftable} \Rightarrow \mathrm{P}_1\text{-liftable}.$$

If ϕ is not path-liftable, the non-liftable index of ϕ is

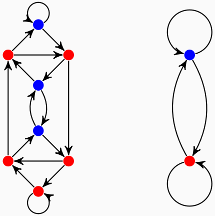
$$\delta(\phi) := \text{the length of shortest walk in } H \text{ without any lifting}$$

An example



A non-liftable 8-walk

An example



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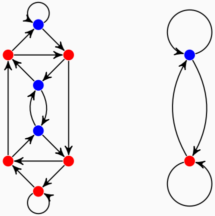

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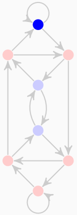

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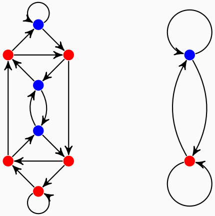

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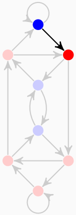

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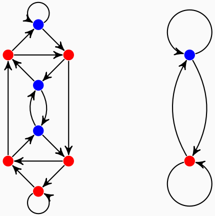

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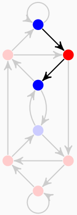

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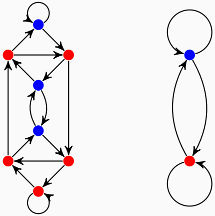

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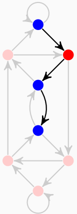

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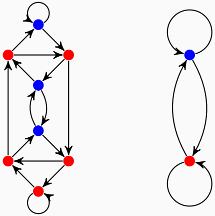

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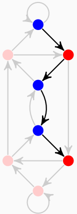

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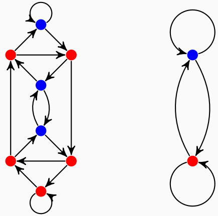

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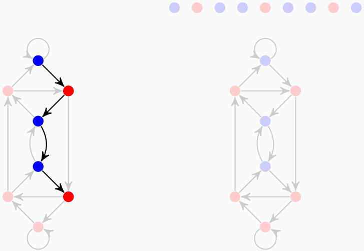

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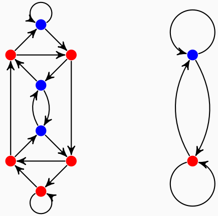

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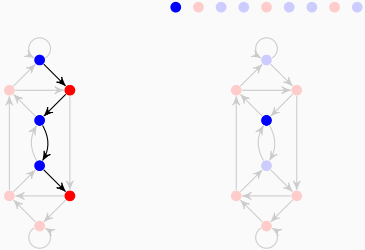

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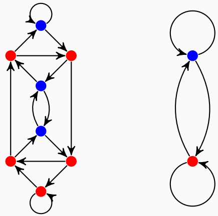

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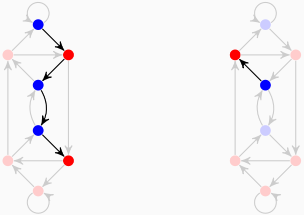

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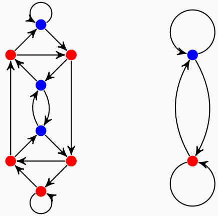

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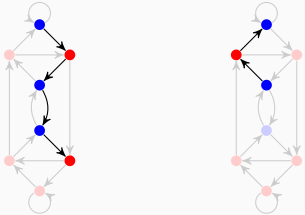

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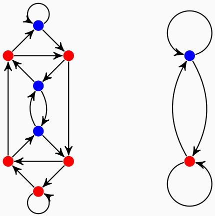

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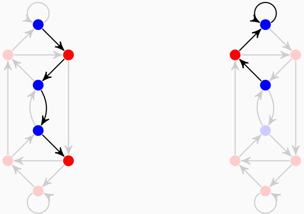

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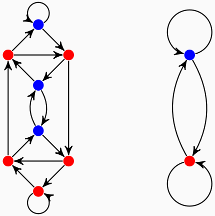

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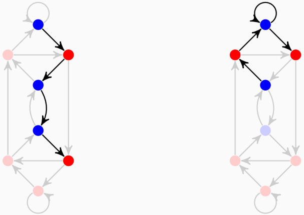

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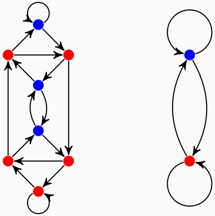

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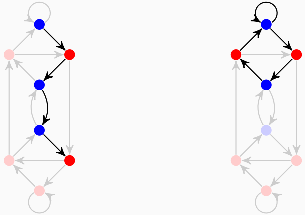

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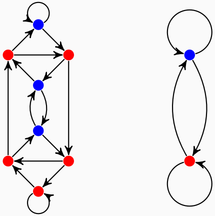

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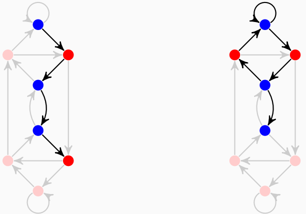

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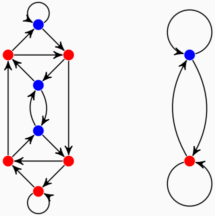

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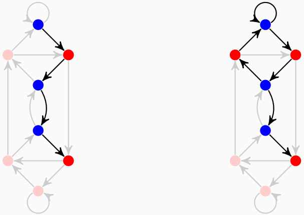

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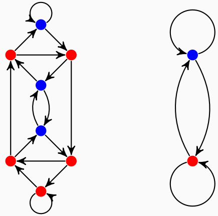

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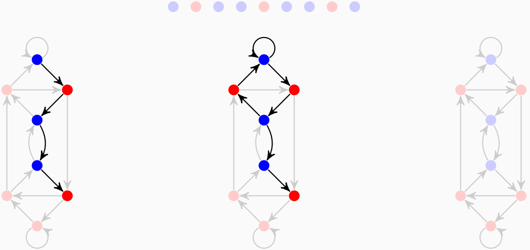

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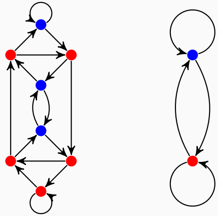

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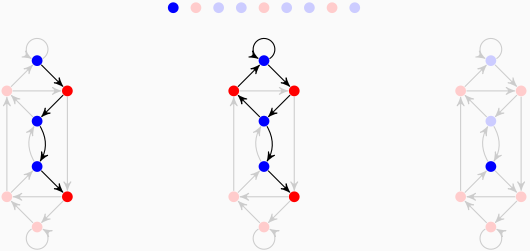

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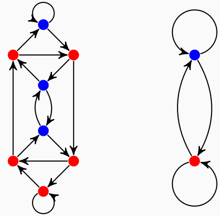

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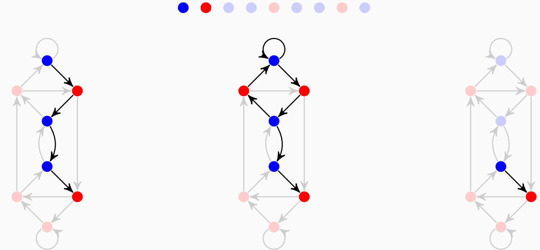


Outline

Question

For given strongly connected digraphs G, H and $\phi : G \to H$, how to detemine whether ϕ is path-liftable or not?

Question

Can we bound $\delta(\phi)$ by the size of G and H?

We will consider the two questions in the following case:

- general case;
- isentropic case, $\lambda_G = \lambda_H$;
- G and H are De Bruijn / Kautz digraphs.

A upper bound of non-liftable indices

For $\phi : G \to H$,

$$\delta(\phi) \leq 2^{|V_G|} - 1.$$

Proof.

- Pick a shortest non-liftable walk $(e_1, e_2, \ldots, e_k)$ in H.
- $\mathcal{R}_0 := \phi_0^{-1}(\mathrm{i}(e_1))$
- $\mathcal{R}_i :=$ {the termial vertices of liftings of $(e_1, \ldots, e_i)$}
- If $\mathcal{R}_i = \mathcal{R}_j$ and $i < j$, then the walk $(e_1, \ldots, e_i, e_{j+1} \ldots, e_k)$ is also a non-liftable walk.
- $\mathcal{R}_0, \mathcal{R}_1 \ldots, \mathcal{R}_k$ are distinct subsets of V_G. Thus $\delta(\phi) = k \leq 2^{|V_G|} - 1$.

□

This bound is tight.

Reverse lexicographical order

Let $\mathbb{B}$ be the Boolean semiring. Let $\prec$ be the reverse lexicographical order on $\mathbb{B}^n$ which is define by $x \prec y$ if $x(i) > y(i)$ for the minimum i where $x(i) \neq y(i)$.

Example

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \prec \begin{pmatrix}1\\1\\0\end{pmatrix} \prec \begin{pmatrix}1\\0\\1\end{pmatrix} \prec \begin{pmatrix}1\\0\\0\end{pmatrix} \prec \begin{pmatrix}0\\1\\1\end{pmatrix} \prec \begin{pmatrix}0\\1\\0\end{pmatrix} \prec \begin{pmatrix}0\\0\\1\end{pmatrix} \prec \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Construction

For $k \in [n]$, define

$$A_k = \begin{array}{c} [k-1] \\ \{k\} \\ [k+1,n] \end{array} \overset{\begin{array}{ccc} [k-1] & \{k\} & [k+1,n] \end{array}}{\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & J \\ J & J & J \end{pmatrix}}.$$

Let $\mathbb{B}^n = \{\pi_i : i \in [2^n]\}$ such that $\pi_i \preceq \pi_j$ if $i \preceq j$. One can check that

$$A_k \pi_i \begin{cases} \preceq \pi_i & \text{if } k \neq p, \\ = \pi_{i+1} & \text{if } k = p. \end{cases}$$

Then $\mathbf{0} \in \langle A_k : k \in [n] \rangle$ and any product of $2^n - 2$ elements in $\{A_k : k \in [n]\}$ is not $\mathbf{0}$.

Construction, cont'd

Construct $\phi : G \to H$ as follow.

- Let H be the digraph such that $V_H = \{1\}$ and $E_H = [n]$.
- Let G be the digraph such that
  - $V_G = [n]$;
  - $E_G = \{(i, j, k) : A_k(i, j) = 1\}$;
  - initial operator is defined by $\mathrm{i}_G\Big((i, j, k)\Big) = i$;
  - terminal operator is defined by $\mathrm{t}_G\Big((i, j, k)\Big) = j$.
- Let $\phi : G \to H$ be the homomorphism such that $\phi_1\Big((i, j, k)\Big) = k$.
- For a walk $(e_1, \ldots, e_k)$ in H,

$$\# \text{ liftings of } (e_1, \ldots, e_k) = \# \text{ 1 in } A_{e_1} A_{e_2} \cdots A_{e_k}.$$

- Then $\delta(\phi) = 2^n - 1 = 2^{|V_G|} - 1$.

A dichotomy

Let H be a fixed strongly connected digraph.

Question

Input a digraph G and $\phi : G \to H$. What is the complexity to determine whether ϕ is path-liftable or not?

If H is a cycle, it is easy (G has non-trivial strongly connected components $\Leftrightarrow$ ϕ is path-liftable).

Theorem (Wu, Z.)

If H is not a cycle, then the determine problem is NP-complete.

- A reduction from 3-SAT problem. **blackboard**

3-CNF formula

Three operators on Boolean semi-field.

x	0	1
$\overline{x}$	1	0

Table: Negation

$\wedge$	0	1
0	0	0
1	0	1

Table: Conjunction

$\vee$	0	1
0	0	1
1	1	1

Table: Disjunction

Let $x_1, \ldots, x_n$ be Boolean variables. A literal is either a variable or the negation of a variable. A clause is the disjunction of three literals. A 3-CNF formula is the conjunction of clauses.

Example

$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee x_4)$

3-SAT problem

For given a 3-CNF formula $\mathcal{F}$, the 3-SAT problem is whether or not an assignment of the variables that make $\mathcal{F} = 1$.

Theorem (Cook, 1971)

3-SAT problem is NP-complete.

Spectral radius

Let G and H be digraphs.

- λ_G: the spectral radius of the adjacency matrix of G.
- Note that

$$\lambda_G = \lim_{k \to +\infty} \frac{1}{k} \log \left(\left| \hom(\mathrm{P}_k, G) \right| \right).$$

- Thus, if $\lambda_G < \lambda_H$, there is no path-liftable homomorphism from G to H.
- What is the phenomenon when $\lambda_G = \lambda_H$?

Diamonds

Let $\phi \in \hom(G, H)$ and $\gamma, \gamma' \in \hom(\mathrm{P}_k, G)$. We call (γ, γ') a diamond of ϕ if

- distinct : $\gamma \neq \gamma'$;
- same image: $\phi \circ \gamma = \phi \circ \gamma'$;
- same initial vertex: $\mathrm{i}(\gamma) = \mathrm{i}(\gamma')$;
- same terminal vertex: $\mathrm{t}(\gamma) = \mathrm{t}(\gamma')$.

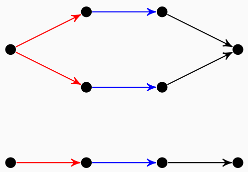


Figure: a diamond

A cubic-time algorithm

Theorem (Well known in symbolic dynamic)

Let G and H be two strongly connected digraphs and $\phi \in \hom(G, H)$, then any two of the following expressions implies the other one.

(1) $\lambda_G = \lambda_H$.

(2) ϕ *is path-liftable.*

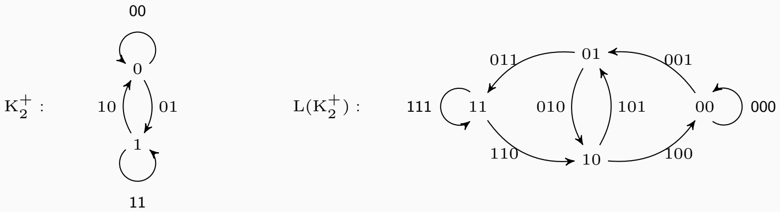
(3) ϕ *has no diamond.*

Theorem (Even[1],1965)

There is an algorithm to determine whether a homomorphism $\phi \in \hom(G, H)$ has a diamond or not in time $O(|\mathrm{V}_G|^3)$

[1] S. Even. "On Information Lossless Automata of Finite Order". In: IEEE Transactions on Electronic Computers EC-14.4 (1965), pp. 561–569.

De Bruijn and Kautz digraphs



- K_n^+: n-vertex complete digraph with loops.
- d-dimension De Bruijn digraph $B(n, d)$: the $(d-1)$-th line digraph of K_n^+.
- K_n: n-vertex complete digraph without loops.
- d-dimension Kautz digraph $K(n, d)$: the $(d-1)$-th line digraph of K_n.

Motivation

Definition (Tvrdik, Harbane and Heydemann[2], 1998)

Let d be an integer, $d \geq 2$. Let $\diamond$ be a binary operation on $\mathbb{Z}_n$ such that for any $y_1, \ldots, y_{d-1} \in \mathbb{Z}_n$, the set of $d-1$ equations

$$x_i \diamond x_{i+1} = y_i, \quad 1 \leq i \leq d-1$$

for unknowns $x_1, \ldots, x_d$ has exactly n distinct solutions such that $x_i \in \mathbb{Z}_n$. Then it is satd that the operation $\diamond$ satisfies Property (P_d).

Their problem is to find all binary operations on $\mathbb{Z}_n$ satisfying Property (P_d) for all d.

[2] Pavel Tvrdik, Rabah Harbane, and Marie-Claude Heydemann. "Uniform homomorphisms of de Bruijn and Kautz networks". In: Discrete Appl. Math. 83.1-3 (1998), pp. 279–301.

Example

$\diamond$	0	1	2	3	4	5
0	5	5	5	1	1	1
1	4	4	4	2	2	2
2	0	0	0	3	3	3
3	3	3	3	2	2	2
4	4	4	4	0	0	0
5	1	1	1	5	5	5

Table: An operation satisfies Property (P_d) for all d.

From operations to digraph homomorphisms

Let $\diamond$ be a binary operation on $\mathbb{Z}_n$

- $\diamond$ is corresponding to the digraph homomorphism $\phi : \mathrm{B}(n, 2) \to \mathrm{B}(n, 1)$ such that $\phi_0(a, b) = a \diamond b$.
- One can show that $\diamond$ satisfies Property (P_d) for all d if and only if ϕ is path-liftable.
- Tvrdik, Harbane and Heydemann also consider a variant definition which is corresponding to the digraph homomorphism from $\mathrm{K}(n, 2)$ to $\mathrm{K}(n, 1)$.

Right-covering (Left-covering) homomorphism

- $\phi \in \hom(G, H)$
- for $v \in V_G$, define $G^+(v) = \{e \in \mathrm{E}_G : \mathrm{i}(e) = v\}$.

ϕ is called right-covering if ϕ_1 is surjective from $G^+(v)$ to $H^+(\phi_0(v))$ for all $v \in \mathrm{V}_G$.
By symmetry, we define left-covering homomorphisms.
We call ϕ a one-sided covering if it is either a right-covering or a left-covering or both.

One-sided covering is always path-liftable.

Conjecture (Tvrdik, Harbane, Heydemann, 1998)

Let n be a prime and let $\phi \in \hom(\mathrm{B}(n,2), \mathrm{B}(n,1))$ and $\psi \in \hom(\mathrm{K}(n+1,2), \mathrm{K}(n+1,1))$.

- *If ϕ is path-liftable, then ϕ is one-sided covering.*
- *If ψ is path-liftable, then ψ is one-sided covering.*
- *If ψ is not path-liftable, then $\delta(\psi) \leq 3$.*

Theorem (Wu,Z.)

Let G and H be two k-regular strongly connected digraphs. If $\frac{|\mathrm{V}_G|}{|\mathrm{V}_H|}$ is a prime number, then ϕ is path-liftable iff it is a one-sided covering.

- Define three positive integer parameters $M(\phi), R(\phi)$ and $L(\phi)$.
- $L(\phi)M(\phi)R(\phi) = \frac{|\mathrm{V}_G|}{|\mathrm{V}_H|} =$ a prime.
- Either $L(\phi)$ or $R(\phi)$ equals 1. Thus ϕ is a one-sided covering.

Degree

- Let G, H be two strongly connected k-regular digraphs.
- A bi-infinite walk $\tau \in \hom(P_\infty, G)$ is doubly transitive if for every finite walk γ in G, it occurs in τ infinite many times in both directions.

$$\tau = \quad \cdots * * * \gamma * * * * \gamma * * * * * * * \gamma * * * * * * \cdots$$

- $\phi \in \hom(G, H)$

Lemma

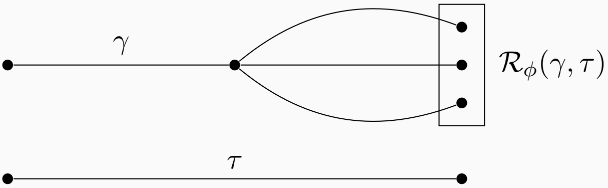
There exists a positive integer $M(\phi)$ such that
$M(\phi) = |\{\alpha \in X_G : \phi \circ \alpha = \tau\}|$ *for all doubly transitive walk τ.*

We call the number $M(\phi)$ the degree of ϕ.

Welch indices

Let G and H be two strongly connected k-regular digraphs. Let $\phi \in \hom(G, H)$ be path-liftable homomorphism. For a finite walk γ in G and a finite walk τ in H, define the ϕ-compatible right extension of (γ, τ) to be

$$\mathcal{R}_\phi(\gamma, \tau) \doteq \{\mathrm{t}(\gamma\gamma') : \phi \circ (\gamma\gamma') = \tau\}.$$



Define $R_\phi(\gamma) = \max_\tau\{|\mathcal{R}_\phi(\gamma, \tau)|\}$ and $L_\phi(\gamma) = \max_\tau\{|\mathcal{L}_\phi(\gamma, \tau)|\}$

Welch indices, cont'd

- [Hedlund[3], 1969] There exists integer $R(\phi)$ and $L(\phi)$ such that $R_\phi(\gamma) = R(\phi)$ and $L_\phi(\gamma) = L(\phi)$ for all γ.
- [Hedlund, 1969] $L(\phi)M(\phi)R(\phi) = \frac{|\mathrm{V}_G|}{|\mathrm{V}_H|}$.
- In Hedlund's paper, $G = \mathrm{B}(n,k)$ and $H = \mathrm{B}(n,k')$. The proof is also vilid for k-regular case.

[3] G. A. Hedlund. "Endomorphisms and automorphisms of the shift dynamical system". In: Math. Systems Theory 3 (1969), pp. 320–375.