Multivariate digraphs and higher order Markov chains

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Digraphs and multivariable digraphs

- Let K be a finite set.
- A (single-variable) digraph f on K is a map from K to 2^K.

- Let t be a positive integer.
- A t-variable digraph (or simply t-digraph) f on K is a map from K^t to 2^K.

Box

- A set of the form
$$\mathcal{A} = A_1 \times A_2 \times \cdots \times A_t$$
where $A_1, \ldots, A_t$ are finite sets, will be called an t-dim discrete box (simply t-box). Write $\mathcal{A}(i)$ for A_i.
- Let $\mathcal{A}$ and $\mathcal{B}$ be t-boxes. We call $\mathcal{B}$ a sub-box of $\mathcal{A}$ if $\mathcal{B}(i) \subseteq \mathcal{A}(i)$ for all $i \in \{1, \ldots, t\}$. We use $\mathrm{BOX}(\mathcal{A})$ to denote the set of sub-boxes of $\mathcal{A}$.
- **Convertion** We identify the t-box $\{a_1\} \times \cdots \times \{a_t\}$ with $(a_1, \ldots, a_t)$.

De Bruijn form and Markov operator

Let f be a t-digraph on K. It can be graphically represented by its De Bruijn form Γ_f, a digraph on K^t such that

$$\Gamma_f(k_1, \ldots, k_t) = \Big\{(k_2, \ldots, k_t, k_{t+1}) \mid k_{t+1} \in f(k_1, \ldots, k_t)\Big\}.$$

The Markov operator M_f associated to a t-digraph f is a map from $\mathrm{BOX}(K^t)$ to $\mathrm{BOX}(K^t)$ such that

$$\mathrm{M}_f(A_1 \times \cdots \times A_t) = A_2 \times \cdots \times A_t \times \Big(\bigcup_{x \in A_1 \times \cdots \times A_t} f(x) \Big).$$

The vertex set of Γ_f is K^t while the vertex set of M_f is $\mathrm{BOX}(K^t)$. Note that $\binom{K}{1}^t \subseteq \mathrm{BOX}(K^t) \subseteq 2^{(K^t)}$ and so $|K^t| \leq (2^{|K|})^t \leq 2^{(|K|^t)}$.

Hitting indices

Let f be a t-digraph on K. For $x, y \in K^t$, we define $\mathcal{HI}_f(x, y)$ to be the set

$$\{n > 0 \mid y \in \mathrm{M}_f^n(x)\}$$

and call it the set of hitting indices of f from x to y.

The distance from x to y in f is defined to be

$$\mathrm{Dist}_f(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \min \mathcal{HI}_f(x, y), & \text{otherwise.} \end{cases}$$

The diameter of f denoted by $\mathrm{Dia}(f)$, is $\max_{x \neq y} \mathrm{Dist}_f(x, y)$.

Convention $\max \emptyset = 0$ and $\min \emptyset = +\infty$.

Strong connectivity

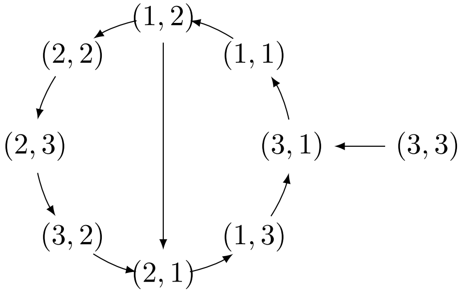
Let K be a finite set and t be a positive integer.

A t-digraph f on K is called strongly connected if $\mathcal{HI}_f(x, y) \neq \emptyset$ for all $x, y \in K^t$.

Note that for a strongly connected t-digraph f, $\mathrm{Dia}(f)$ is always smaller than $(2^{|K|})^t$ but may be larger than the size of the vertex set of Γ_f, that is $|K|^t$.

An example

$K = \{1, 2, 3\}$
$\mathrm{Dia}(f) = \mathrm{Dist}_f((2,2),(3,3)) = 15 > 3^2.$



Γ_f

A question

What is the maximum value of the diameter of a strongly connected t-digraph?

Maximum diameter of 2-digraph

Let k be a positive integer and let $[k] = \{1, 2, \ldots, k\}$.
$\mathrm{D}_{t,k} := \max \{\mathrm{Dia}(f) \mid f$ is a strongly connected t-digraph on $[k]\}$.

- $\mathrm{D}_{2,2} = 4$ and $\mathrm{D}_{2,3} = 15$.

Theorem (Wu, Xu, Z., 2017)

For $k \geq 5$,

$$\mathrm{D}_{2,k} \geq \begin{cases} 2k^2, & \text{if } k \text{ is odd;} \\ 2k^2 - k + 1, & \text{if } k \text{ is even.} \end{cases}$$

Conjecture (Wu, Xu, Z., 2017)

$$\lim_{k\to\infty} \frac{\mathrm{D}_{2,k}}{k^2} = 2.$$

Period classes

Let f be a strongly connected t-digraph on a finite set K.

Define an equivalence relation $\sim_f$ on K^t, which is for all $x, y \in K^t$, $x \sim_f y$ if and only if there exists a positive integer m such that $\mathrm{M}_f^m(x) = \mathrm{M}_f^m(y)$. Each equivalent class $\sim_f$ is called a periodic class of f. We define the period of f to be

$$\mathrm{per}(f) := \text{the number of periodic classes of } f.$$

Lemma (Wu, Xu, Z.)

$\mathrm{per}(f) = \gcd\Big(\mathcal{HI}_f(x,x)\Big)$ *for all* $x \in K^t$.

Cyclic decomposition

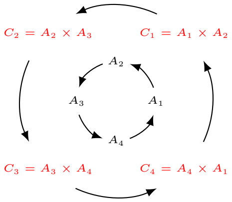
Theorem (Wu,Xu,Z.,2017)

Let f be a strongly connected t-digraph on K with $\mathrm{per}(f) = p$. Then the periodic classes of f can be enumerated as $\{C_i\}_{i\in\mathbb{Z}_p}$ such that the following hold:

1. $C_i \in \mathrm{BOX}(K^t)$ *and* $\mathrm{M}_f(C_i) = C_{i+1}$ *for all* $i \in \mathbb{Z}_p$;
2. *there is a set* $\{A_i \mid A_i \subseteq K\}_{i\in\mathbb{Z}_p}$ *such that*

$$C_i = A_i \times A_{i+1} \times \cdots \times A_{i+t-1}$$

for all $i \in \mathbb{Z}_p$.



Set of periods

- Let t be a positive integer.
- $\mathrm{per}(t) := \{\mathrm{per}(f) \mid f$ is a strongly connected t-digraph$\}$.

Question

Let t be a positive integer. Determine the set $\mathrm{per}(t)$.

- $\mathrm{per}(1) = \mathbb{Z}^+$;
- $\mathrm{per}(2) = \mathbb{Z}^+ \setminus \{2, 3, 5, 6, 7\} \supsetneq \mathrm{per}(3)$.

Theorem (Wu, Xu, Z., 2017)

For any fixed t, there are only finitely many positive integers outside of $\mathrm{per}(t)$.

Lower bound and upper bound of $\alpha(t)$

$$\alpha(t) = \min\left\{n \in \mathbb{Z}^+ \mid \mathrm{per}(t) \supseteq \{n, n+1, \ldots\}\right\}.$$

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For a proof of the above theorem, we need to use the following result about t-box.

Theorem (Alon, Bohman, Holzman, Kleitman, 2002)

Let $\mathcal{A}$ be a t-box and let $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p\}$ be a partition of $\mathcal{A}$ such that $\emptyset \neq \mathcal{B}_j(i) \neq \mathcal{A}(i)$ holds for all i, j. Then $p \geq 2^t$.

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Theorem (Qian, Wu, Z., 2018+)

For sufficently large t, $\alpha(t) \leq (\frac{t^2}{2})^t$.

The second minimum period

Note that $1 \in \mathrm{per}(t)$ for every positive integer t.

Question

What is the second minimum period of strongly connected t-digraphs?

Theorem (Qian, Wu, Z., 2018+)

$\min(\mathrm{per}(t) \setminus \{1\}) \geq 2^{\lceil \frac{t}{2} \rceil}$.

Two conjectures of periods

Conjecture (Wu, Xu, Z., 2017)

$\mathrm{per}(t+1) \subsetneq \mathrm{per}(t)$.

Conjecture (Qian, Wu, Z., 2018+)

For sufficently large t, the second minimum element of $\mathrm{per}(t)$ is larger than $\lceil 2^{\frac{t}{2}} \rceil$.

THANK YOU FOR YOUR ATTENTION.