Hurwitz primitivity and synchronizing automata

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Primitivity

A nonnegative n-by-n matrix A is called **primitive** if $A^k > 0$ (entrywise) for some $k \geq 0$.

There are several possibilities to generalize the concept "primitivity" from a nonnegative matrix (Markov process) to a tuple of nonnegative matrices.

Today, we focus on two generalizations:

- primitivity (inhomogeneous Markov process)
- Hurwitz primitivity (multi-dimensional Markov process)

If a process is (Hurwitz) primitive, it has some nice asymptotic behavior.

Primitive matrix tuples

Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an m-tuple of nonnegative n-by-n matrices. For each finite sequence $\alpha = \alpha_1 \cdots \alpha_k$ over $[m] = \{1, 2 \ldots, m\}$, write $\mathcal{A}_\alpha$ for $A_{\alpha_1} \cdots A_{\alpha_k}$ and call it a product over $\mathcal{A}$ of length k.

- The m-tuple $\mathcal{A}$ is called **primitive** if there exists a finite sequence α over $[m]$ such that
$$\mathcal{A}_\alpha > 0.$$
- The minimum length of positive products over $\mathcal{A}$ is called the **primitive index** of $\mathcal{A}$.

Types of sequences

Let $\alpha = \alpha_1 \cdots \alpha_k$ be a sequence over a set X.

- For any $x \in X$, we denote the number of **occurrences** of x in the word α by $|\alpha|_x$, that is
$$|\alpha|_x = |\{i \in [k] : \alpha_i = x\}|.$$
- The **type** of α, denoted by $\mathrm{t}(\alpha)$, is the vector in $\mathbb{N}^X$ such that
$$\mathrm{t}(\alpha)(x) = |\alpha|_x$$
for each $x \in X$.

Example

The type of the sequence $\alpha = 1442112$ over $\{1, 2, 3, 4\}$ is
$$\mathrm{t}(\alpha) = (3, 2, 0, 2).$$

Hurwitz products and Hurwitz primitivity

Let $\mathcal{A} = (A_1, \ldots, A_m)$ an m-tuple of nonnegative n-by-n matrices. For each $\tau = (\tau_1, \ldots, \tau_m) \in \mathbb{N}^m$, let

$$\mathcal{A}^\tau = \sum_{\alpha:\ \mathrm{t}(\alpha)=\tau} \mathcal{A}_\alpha \,.$$

We call $\mathcal{A}^\tau$ a **Hurwitz product** of $\mathcal{A}$ of length $|\tau| := \sum_{i=1}^m \tau_i$.

- The tuple $\mathcal{A}$ is **Hurwitz primitive** if it owns a positive Hurwitz product.
- The minimum length of positive Hurwitz products is called the **Hurwitz primitive index** of $\mathcal{A}$.

Example

- $\mathcal{A} = (A_1, A_2, A_3)$.
- $\mathcal{A}^{(1,3,0)} = A_1 A_2^3 + A_2 A_1 A_2^2 + A_2^2 A_1 A_2 + A_2^3 A_1$.

Problems

- For a matrix tuple, how to determine whether it is (Hurwitz) primitive or not?
- For a (Hurwitz) primitive matrix tuple, how to find a positive (Hurwitz) product of it?
- What is the maximum (Hurwitz) primitive index of all (Hurwitz) primitive m-tuples of n-by-n nonnegative matrices?

Determine Problems

- [Gerencsér-Gusev-Jungers[1], 2018] The determine problem of primitivity is NP-hard (even for two matrices).
- The algorithmic complexity of determining Hurwitz primitivity is still unknown.

[1] Balázs Gerencsér, Vladimir V. Gusev, and Raphaël M. Jungers (2018). "Primitive sets of nonnegative matrices and synchronizing automata". In: *SIAM J. Matrix Anal. Appl.* 39.1, pp. 83–98.

Two subfamilies of square matrices

- The set of nonnegative n-by-n matrices that has no zero rows is denoted by $\mathrm{NZ}_1(n)$. (row-stochastic matrix)
- The set of nonnegative n-by-n matrices that has no zero rows and no zero columns is denoted by $\mathrm{NZ}_2(n)$. (doubly-stochastic matrix)

Block permutation matrices

Let A be an n-by-n matrix. Let $\pi = (\pi_1, \ldots, \pi_r)$ be a partition of $[n]$. We say that A **preserves the partition** π if there exists a permutation $\sigma \in \mathrm{Sym}_r$ such that $A(\pi_i, \pi_j) = 0$ whenever $j \neq \sigma(i)$.

Two characterization theorems

- A tuple of nonnegative matrices $\mathcal{A}$ is **irreducible** if $\sum_{A \in \mathcal{A}} A$ is irreducible.
- A partition is **trivial** if it contains at least two parts.

Theorem (Protasov-Voynov[2], 2012)

Let $\mathcal{A}$ be an irreducible tuple of NZ_2-matrices. The tuple $\mathcal{A}$ is not primitive if and only if there exists a non-trivial partition π such that every matrix in $\mathcal{A}$ preserves π.

Theorem (Protasov[3], 2013)

Let $\mathcal{A}$ be an irreducible tuple of NZ_1-matrices. The tuple $\mathcal{A}$ is not Hurwitz primitive if and only if there exists a non-trivial partition π such that every matrix in $\mathcal{A}$ preserves π and all these permutations corresponding to members of $\mathcal{A}$ commute with each other.

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[2] V.Yu. Protasov and A.S. Voynov (2012). "Sets of nonnegative matrices without positive products". In: *Linear Algebra and its Applications* 437.3, pp. 749–765.

[3] V.Yu. Protasov (2013). "Classification of k-primitive sets of matrices". In: *SIAM J. Matrix Anal.* 34.3, pp. 1174–1188.

Different proofs

Characterization theorem of primitive matrices in $\mathrm{NZ}_2(n)$:

- Protasov-Voynov (2012) give the first proof by using geometrical properties of affine operators on polyhedra.
- Three combinatorial proofs are found by Al'pin-Alpina (2013), Blondel-Jungers-Olshevsky (2015), and Al'pin-Alpina (2019).
- Using analytic method, Protasov (2021) gives a new proof.

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Characterization theorem of Hurwitz primitive matrices in $NZ_1(n)$:

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We will present a sketch of a unified proof of these two characterization theorems.

A sketch of the proof (primitive)

Let $\mathcal{A}$ be a m-tuple of nonnegative n-by-n NZ_2-matrices.
Define $\approx$ to be the binary relation on $[n]$ such that $i \approx j$ if for all $i', j' \in [n]$ and for all finite sequence α over $[m]$ satisfying

$$\mathcal{A}_\alpha(i, i') > 0 \quad \text{and} \quad \mathcal{A}_\alpha(j, j') > 0,$$

there exists $k \in [n]$ and a sequence β such that

$$\mathcal{A}_\beta(i', k) > 0 \quad \text{and} \quad \mathcal{A}_\beta(j', k) > 0.$$

The relation $\approx$ is called the **stable relation** of $\mathcal{A}$.

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The relation $\approx$ is called the **stable relation** of $\mathcal{A}$. It is routine to verify the following statements.

- The relation $\approx$ is an equivalence relation.
- Let π be the partition which is formed by the equivalence class of $\approx$. The matrices in $\mathcal{A}$ preserve π.
- The partition π is the unique minimal (finest) partition of $[n]$ such that all matrices in $\mathcal{A}$ preserve it.

A sketch of the proof (Hurwitz primitive)

Let $\mathcal{A}$ be a m-tuple of nonnegative n-by-n NZ_1-matrices.

Define $\overset{\mathsf{h}}{\approx}$ to be the binary relation on $[n]$ such that $i \overset{\mathsf{h}}{\approx} j$ if for all $i', j' \in [n]$ and for all vector $\tau \in \mathbb{N}^m$ satisfying

$$\mathcal{A}^{\tau}(i, i') > 0 \quad \text{and} \quad \mathcal{A}^{\tau}(j, j') > 0,$$

there exists $k \in [n]$ and a vector $\beta \in \mathbb{N}^m$ such that

$$\mathcal{A}^{\gamma}(i', k) > 0 \quad \text{and} \quad \mathcal{A}^{\gamma}(j', k) > 0.$$

The relation $\overset{\mathsf{h}}{\approx}$ is called the **Hurwitz stable relation** of $\mathcal{A}$. It is routine to verify the following statements.

- The relation $\overset{\mathsf{h}}{\approx}$ is an equivalence relation.
- Let π be the partition which is formed by the equivalence class of $\overset{\mathsf{h}}{\approx}$. The matrices in $\mathcal{A}$ preserve π.
- The partition π is the unique minimal (finest) partition of $[n]$ such that all matrices in $\mathcal{A}$ preserve π and all these permutations corresponding to members of $\mathcal{A}$ commute with each other.

Maximum (Hurwitz) primitive index

Let X be a subfamily of nonnegative matrices.

- $\mathrm{p}_X(n) \doteq$ the maximum primitive index of all primitive tuples of n-by-n X-matrices;
- $\mathrm{hp}_X(n) \doteq$ the maximum Hurwitz primitive index of all Hurwitz primitive tuples of n-by-n X-matrices.

We will present some results on $\mathrm{p}_{\mathrm{NZ}_2}(n)$ and $\mathrm{hp}_{\mathrm{NZ}_1}(n)$.

$\mathrm{p}_{\mathrm{NZ}_2}(n)$ and $\mathrm{hp}_{\mathrm{NZ}_1}(n)$

- [Blondel-Jungers-Olshevsky[4], 2015]

$$\frac{n^2}{2} \leq \mathrm{p}_{\mathrm{NZ}_2}(n) \leq \frac{n^3 + 2n - 3}{3}$$

- [Gusev[5], 2013]

$$(n-1)^2 \leq \mathrm{hp}_{\mathrm{NZ}_1}(n)$$

[4] Vincent D. Blondel, Raphaël M. Jungers, and Alex Olshevsky (2015). "On primitivity of sets of matrices". In: *Automatica J. IFAC* 61, pp. 80–88.

[5] Vladimir V. Gusev (2013). "Lower bounds for the length of reset words in Eulerian automata". In: *Internat. J. Found. Comput. Sci.* 24.2, pp. 251–262.

$\mathrm{p}_{\mathrm{NZ}_2}(n)$ and $\mathrm{hp}_{\mathrm{NZ}_1}(n)$

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$$\frac{n^2}{2} \leq \mathrm{p}_{\mathrm{NZ}_2}(n) \leq \frac{n^3 + 2n - 3}{3}$$

- [Gusev[5], 2013; Wu-Z., 2022+]

$$(n-1)^2 \leq \mathrm{hp}_{\mathrm{NZ}_1}(n) \leq 2\,\mathrm{c}(n) + \left\lfloor \frac{(n+1)^2}{4} \right\rfloor = O(n^3)$$

[4] Vincent D. Blondel, Raphaël M. Jungers, and Alex Olshevsky (2015). "On primitivity of sets of matrices". In: *Automatica J. IFAC* 61, pp. 80–88.

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Synchronizing automata

- A square $(0,1)$-matrix is called an **automaton matrix** if each row of A contains a unique 1.
- An n-state **automaton** is a tuple of n-by-n automaton matrices.
- An automaton $\mathcal{A}$ is **synchronizing** if there exists a product $\mathcal{A}_\alpha$ which contains a positive column.
- The minimum length of such products is called **synchronizing index** of $\mathcal{A}$.

Černý Conjecture

Define the **Černý function** $c(n)$ as the maximum synchronizing index of all synchronizing automata with n states.

Conjecture (Černý, 1971[6])

$c(n) = (n-1)^2$.

[6] Ján Černý, Alica Pirická, and Blanka Rosenauerová (1971). "On directable automata". In: *Kybernetika (Prague)* 7, pp. 289–298. ISSN: 0023-5954.

Some progresses on Černý Conjecture

In 1964, Černý[7] found a family of automata $\{\mathcal{C}_n\}$ such that $\mathcal{C}_n$ is an n-state synchronizing automaton whose synchronizing index equals $(n-1)^2$. This shows that

$$(n-1)^2 \leq \mathrm{c}(n).$$

[7] Ján Černý (1964). "A remark on homogeneous experiments with finite automata". In: *Mat.-Fyz. Časopis. Sloven. Akad. Vied.* 14. (Slovak. English summary), pp. 208–216. ISSN: 0543-0046.

Some progresses on Černý Conjecture, Cont'd

There are some upper bounds of $c(n)$ which roughly equals $O(\frac{n^3}{6})$.

- [Frankl[8]-Pin[9] 1982] $c(n) \leq \frac{n^3-n}{6} \leq O(0.16667n^3)$
- [Szykuła[10] 2018] $c(n) \leq \frac{85059n^3+90024n^2+196504n-10648}{511104} \leq O(0.16643n^3)$
- [Shitov[11] 2019] $c(n) \leq \left(\frac{7}{48} + \frac{15625}{798768}\right) n^3 + o(n^2) \leq O(0.16540n^3)$

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[8] P. Frankl (1982). "An extremal problem for two families of sets". In: *European J. Combin.* 3.2, pp. 125–127.

[9] J.-E. Pin (1983). "On two combinatorial problems arising from automata theory". In: *Combinatorial mathematics (Marseille-Luminy, 1981)*. Vol. 75. North-Holland Math. Stud. Pp. 535–548.

[10] Marek Szykuła (2018). "Improving the upper bound and the length of the shortest reset words". In: vol. 96. LIPIcs. Leibniz Int. Proc. Inform. Art. No. 56, 13.

[11] Y. Shitov (2019). "An improvement to a recent upper bound for synchronizing words of finite automata". In: *Journal of Automata, Languages and Combinatorics* 24, pp. 367–373.

Connection between Hurwitz primitive NZ_1-matrix tuples and Synchronizing Automata

Let $\mathcal{A}$ be an Hurwitz primitive tuple of n-by-n NZ_1-matrix. Without loss of genernality, we can assume matrices in $\mathcal{A}$ are $(0,1)$-matrices.

- $\mathcal{B} \doteq \mathcal{A} \cup \{A_iA_j + A_jA_i : A_i, A_j \in \mathcal{A}\}$.
- $\mathcal{C} \doteq \{C : C \leq B \in \mathcal{B}$ and C is an automaton matrix$\}$.

Observation (Wu-Z., 2022+)

The automaton $\mathcal{C}$ is synchronizing.

Proof of the upper bound of $\mathrm{hp}_{\mathrm{NZ}_1}(n)$

- Regard $\mathcal{A} = (A_1, \ldots, A_m)$ as an arc-labeled digraph D, where $V(D) = [n]$ and $E(D) = \{x \xrightarrow{k} y : A_k(x, y) > 0\}$.

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- Find a positive Hurwitz product of $\mathcal{A} \Leftrightarrow$ find $\tau \in \mathbb{N}^m$ such that for all vertices x and y there exists a walk from x to y such that the arc-label sequence of this walk is type-τ.

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- By the observation in the last slides, there exists $\tau' \in \mathbb{N}^m$ and a vertex z such that for each vertex x there exists a walk from x to z satisfying the arc-label sequence of this walk is type-τ' and $|\tau'| \leq 2\,\mathrm{c}(n)$.

Proof of the upper bound of $\mathrm{hp}_{\mathrm{NZ}_1}(n)$

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- Since the digraph D is strongly connected, there exists a closed walk W which visits every vertex and has length at most $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$.

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- Since the digraph D is strongly connected, there exists a closed walk W which visits every vertex and has length at most $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$.
- For all vertices x and y, we "connect" W and one of τ'-walks in a proper way to construct a walk from x to y.

Summary

	Primitive		Hurwitz Primitive	
Assumption		NZ_2		NZ_1
Determine problem	NP-hard	$O(n^2 m)$	?	$O(n^2 m^2 + n^3 m)$
Finding such a product	NP-hard	$O(n^3 m)$	?	$O(n^3 m^2)$
Finding such a shortest product	NP-hard	NP-hard	?	?
Lower bounds of indices	$3^{\frac{n}{3}(1-\epsilon)}$	$\frac{n^2}{2}$	Cn^{m+1}	$(n-1)^2+1$
Upper bounds of indices	$3^{\frac{n}{3}(1+\epsilon)}$	$O(n^3)$	$m!mn^{m+1}+n^2$	$O(n^3)$

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THANK YOU FOR YOUR ATTENTION