Combinatorics of Discrete Dynamical Systems

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Higher order inhomogenous DDS

Let t be a positive integer. Let S be a set and let $\mathcal{F}$ be a collection of t-variable maps from S^t to S. The pair $(S, \mathcal{F})$ is called an order-t discrete dynamical system (t-DDS).

- initial states: $x = (x_1, x_2, \ldots, x_t) \in S^t$;
- dynamical mechanism: $F = (f_1, f_2, \ldots) \in \mathcal{F}^{\mathbb{N}}$;
- a trajectory determined by (x, F): $\mathbb{X} = (x_1, x_2, \ldots) \in S^{\mathbb{N}}$, where

$$x_{i+t} = f_i(x_i, x_{i+1}, \ldots, x_{i+t-1}), \forall i \in \mathbb{N}\,.$$

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$$x_{i+t} = f_i(x_i, x_{i+1}, \ldots, x_{i+t-1}), \forall i \in \mathbb{N}.$$

The phase space of $(S, \mathcal{F})$ is an arc-labelled digraph with

- vertex set S^t;
- arc set $\{(x_1, \ldots, x_t) \xrightarrow{f} (x_2, \ldots, x_t, f(x_1, \ldots, x_t)) : x_i \in S, f \in \mathcal{F}\}$,

and will be denoted by $\mathcal{PS}_{\mathcal{F}}$. One-sided infinite walks in the phase space are just all trajectories in $(S, \mathcal{F})$.

Morkov operator

Let $t \in \mathbb{N}$ and K be a finite set. Write $\mathrm{Set}_K := 2^K \setminus \{\emptyset\}$. Let f be a map from K^t to Set_K. It can be graphically represented by its De Bruijn form Γ_f, a digraph with vertex set K^t and arc set

$$\{(x_1, \ldots, x_t) \to (x_2, \ldots, x_{t+1}) : x_{t+1} \in f(x_1, \ldots, x_t)\}.$$

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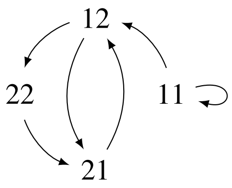
The map f induces a map M_f from Set_K^t to Set_K, called the Morkov operator associated to f, such that

$$\mathrm{M}_f(A_1, \dots, A_t) = \bigcup_{x \in A_1 \times \cdots \times A_t} f(x).$$

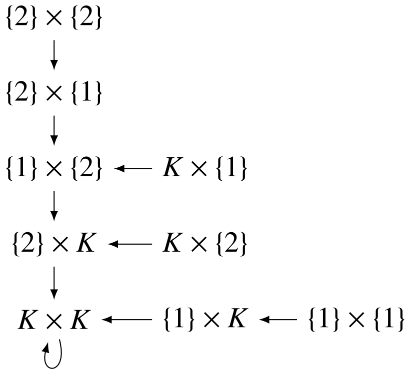
Let $\mathcal{F}$ be a finite collection of maps from K^t to Set_K. We define $\mathrm{M}_{\mathcal{F}}$ to be the multi-set $\{\mathrm{M}_f : f \in \mathcal{F}\}$.

Example

$t = 2, K = \{1, 2\}$



Γ_f



$\mathcal{PS}_{\mathrm{M}_f}$

Figure: The De Bruijn form Γ_f and the phase space $\mathcal{PS}_{\mathrm{M}_f}$.

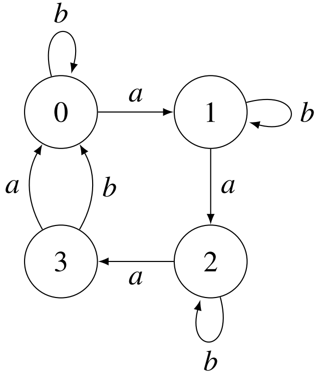
(t, r, k)-EBDDS

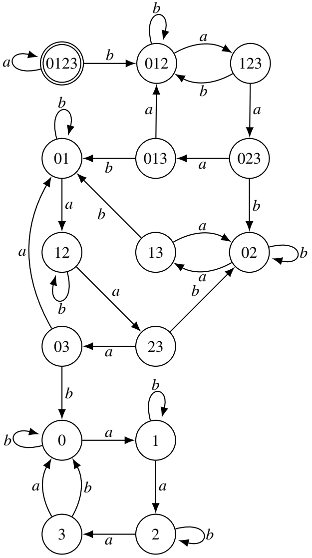
Let $t, r, k \in \mathbb{N}$. Let K be a set of size k and let $\mathcal{F}$ be a collection of r maps from K^t to Set_K. The pair $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$ is called a (t, r, k)-essential Boolean-linear discrete dynamincal systems or, simply, a (t, r, k)-EBDDS.

Convention:

- In the case that $\mathcal{F} = \{f\}$, we write $(\mathrm{Set}_K, \mathrm{M}_f)$ for $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$.
- For $a = (a_1, \ldots, a_t) \in K^t$, we write $\{a\}$ for $\{a_1\} \times \cdots \times \{a_t\}$.

Černý automata as a $(1, 2, k)$-EBDDS





Černý Conjecture

Let $r, k \in \mathbb{N}$. Let K be a set of size k and let $\mathcal{F}$ be a set of r maps from K to Set_K.

If every element from $\mathcal{F}$ is indeed a map from K to $\binom{K}{1}$ and there exists a walk from K to a singleton set in $\mathcal{PS}_{\mathrm{M}_{\mathcal{F}}}$, we call the $(1, r, k)$-EBDDS $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$ a synchronizing automaton.

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Conjecture (Černý, 1964)

Let $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$ be a synchronizing $(1, r, k)$-EBDDS. Then there exists a walk W from K to a singleton set in $\mathcal{PS}_{\mathrm{M}_{\mathcal{F}}}$ with length at most $(k-1)^2$.

Primitive exponent

A (t,r,k)-EBDDS $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$ is primitive if every long enough walk in $\mathcal{PS}_{\mathrm{M}_{\mathcal{F}}}$ ends at K^t.

For a primitive (t,r,k)-EBDDS $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$, the primitive exponent is the minimum non-negative integer T such that every length-T walk in $\mathcal{PS}_{\mathrm{M}_{\mathcal{F}}}$ ends at K^t, which we denote by $\mathrm{g}(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$.

We use $\gamma(t,r,k)$ to denote the maximum number in the set

$$\{\,\mathrm{g}(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}}) : \ (\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}}) \text{ is a primitive } (t,r,k)\text{-EBDDS}\}.$$

Maximum primitive exponents of $(1, r, k)$-EBDDS

Theorem (Wielandt, 1959)

$$\gamma(1,1,k) = \begin{cases} (k-1)^2+1, & \textit{if } k \geq 2; \\ (k-1)^2 = 0, & \textit{if } k = 1. \end{cases}$$

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$$\gamma(1, 2^k - 2, k) = 2^k - 2,$$

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Let $r(k)$ be minimum positive integer r such that

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$$\gamma(1, 2^k-2, k) = 2^k - 2,$$

namely,

$$r(k) \leq 2^k - 2.$$

Maximum primitive exponents of $(1, r, k)$-EBDDS, cont'd

Theorem (Wu, Z., 2015)

$$\gamma(1, k, k) = 2^k - 2,$$

or, equivalently,

$$r(k) \leq k.$$

Note that

$$\gamma(1, 1, k) \leq \gamma(1, 2, k) \leq \cdots \\ \leq \gamma(1, r(k) - 1, k) < \gamma(1, r(k), k) = \gamma(1, r(k) + 1, k) = \cdots = 2^k - 2.$$

Question

Does it hold that $\gamma(1, i, k) < \gamma(1, i + 1, k)$ for all $i \in [1, r(k) - 1]$?

In the spirit of Černý

Conjecture (Wu, Xu, Z., 2016)

Let $(\mathrm{Set}_K, \mathrm{M}_{\mathcal{F}})$ *be a primitive* $(1, r, k)$*-EBDDS. For any* $A, B \in \mathrm{Set}_K$*, it holds*

$$\mathrm{DIST}_{\mathcal{PS}_{\mathrm{M}_{\mathcal{F}}}}(A, B) \leq |B|k^r$$

as long as A *can reach* B *in* $\mathcal{PS}_{\mathrm{M}_{\mathcal{F}}}$*.*

Remark

If this conjecture is correct, then for every $k \geq 2$ *we can obtain*

$$r(k) \geq \lceil \frac{\log_2(2^k - 2)}{\log_2 k} \rceil - 1.$$

Maximum primitive exponents of $(t, 1, k)$-EBDDS

- $\gamma(2, 1, 1) = 1$.
- $\gamma(2, 1, 2) = 7$.
- $\gamma(2, 1, 3) = 23$.

Theorem (Wu, Xu, Z., 2016+)

For $k \geq 4$,

$$\gamma(2, 1, k) \geq (2k - 1)^2 + 1. \quad (1)$$

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Theorem (Chen, Wu, 2016+)

$$k^t \leq \gamma(t, 1, k) \leq t(k - 1)(k^t - 1) + 1.$$

Strong connectivity of $(t, 1, k)$-EBDDS

Let $(\mathrm{Set}_K, \mathrm{M}_f)$ be a $(t, 1, k)$-EBDDS.

For $a, b \in K^t$, we define $\mathcal{RI}_f(a, b)$ to be the set

$$\{i \geq 0 : \ b \in \mathrm{M}_f^i(\{a\})\}$$

and call it the set of reachable indices of f from a to b.

We call $(\mathrm{Set}_K, \mathrm{M}_f)$ strongly connected if $\mathcal{RI}_f(a, b) \neq \emptyset$ for all $a, b \in K^t$.

Diameter of $(t, 1, k)$-EBDDS

Let $(\mathrm{Set}_K, \mathrm{M}_f)$ be a strongly connected $(t, 1, k)$-EBDDS.
For all $a, b \in K^t$, we define the distance from a to b:

$$\mathrm{DIST}_f(a, b) = \min\left\{i : \ i \in \mathcal{RI}_f(a, b)\right\}.$$

The diameter of $(\mathrm{Set}_K, \mathrm{M}_f)$:

$$\mathrm{DIA}(f) = \max\left\{\mathrm{DIST}_f(a, b) : \ a, b \in K^t\right\}.$$

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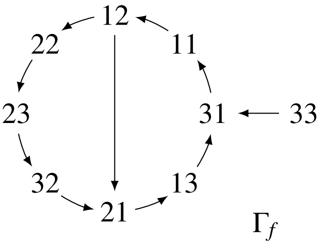
Note that $\mathrm{DIA}(f)$ may be larger than k^t when $t \geq 2$.



$$\mathrm{DIA}(f) = \mathrm{DIST}_f((2, 2), (3, 3)) = 15 > 3^2.$$

$$\mathbb{X} = (\{2\}, \{2\}, \{3\}, \{2\}, \{1\}, \{3\}, \{1\}, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 2\}, \{1\}, \{2, 3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \cdots)$$

Maximum diameter of $(t, 1, k)$-EBDDS

- $\mathrm{D}_{t,k} :=$
  $\max\left\{\mathrm{DIA}(f) : (\mathrm{Set}_K, \mathrm{M}_f) \text{ is a strongly connected } (t,1,k)\text{-EBDDS}\right\}.$

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- $\mathrm{D}_{2,2} = 4$ and $\mathrm{D}_{2,3} = 15$.

Theorem (Wu, Xu, Z., 2016+)

For $k \geq 5$,

$$\mathrm{D}_{2,k} \geq \begin{cases} 2k^2, & \text{if } k \text{ is odd,} \\ 2k^2 - k + 1, & \text{if } k \text{ is even.} \end{cases}$$

Conjecture (Wu, Xu, Z., 2016+)

$$\lim_{k\to\infty} \frac{\mathrm{D}_{2,k}}{k^2} = 2.$$

Cyclic classes

Let $(\mathrm{Set}_K, \mathrm{M}_f)$ be a strongly connected $(t, 1, k)$-EBDDS. We say $a, b \in K^t$ are equivalent in the relation $\sim_{\mathrm{M}_f}$ if there exists $N > 0$ such that

$$\mathrm{M}_f^N(\{a\}) = \mathrm{M}_f^N(\{b\}).$$

Let $C_f := K^t / \sim_{\mathrm{M}_f}$. Each equivalent class of $\sim_{\mathrm{M}_f}$ is called a cyclic class of $(\mathrm{Set}_K, \mathrm{M}_f)$. We define the period of $(\mathrm{Set}_K, \mathrm{M}_f)$ to be the number of equivalent classes and denote it by $\mathrm{per}(f)$.

Cyclic classes, Cont'd

Theorem (Wu, Xu, Z., 2016+)

Let $(\mathrm{Set}_K, \mathrm{M}_f)$ *be a strongly connected* $(t, 1, k)$*-EBDDS. Then the following hold:*

(i) $\mathrm{per}(f) = \gcd(\mathcal{RI}_f(a, a))$ *for all* $a \in K^t$.

(ii) *Let* $\overline{\mathrm{M}_f} : C_f \to C_f$ *such that*

$$\overline{\mathrm{M}_f}([a]) = [b],$$

where $b \in \mathrm{M}_f(\{a\})$ *and* $a, b \in K^t$. *Then the transformation semigroup generated by* $\overline{\mathrm{M}_f}$ *is the cyclic group* $\mathbb{Z}_{\mathrm{per}(f)}$.

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Theorem (Wu, Xu, Z., 2016+)

Let $(\mathrm{Set}_K, \mathrm{M}_f)$ *be a* $(t, 1, k)$*-EBDDS. Then* $(\mathrm{Set}_K, \mathrm{M}_f)$ *is primitive if and only if* $(\mathrm{Set}_K, \mathrm{M}_f)$ *is strongly connected and has period one.*

Sets of periods

Let $\mathcal{P}(t,k)$ be the set of positive integers which can be the period of a strongly connected $(t,1,k)$-EBDDS and let $\mathcal{P}(t) = \bigcup_{k\in\mathbb{N}} \mathcal{P}(t,k)$.

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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t \geq 5$
$\mathcal{P}(t) \cap [1, 2t+3]$	$[1, 2t+3]$	$\{1, 2t\}$	$\{1, 2t+2\}$	$\{1, 2t\}$	$\{1\}$

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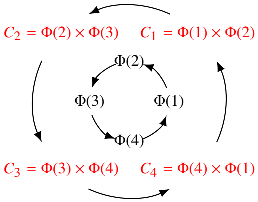
Conjecture (Wu, Xu, Z., 2016+)

$$\mathcal{P}(1) \supsetneqq \mathcal{P}(2) \supsetneqq \mathcal{P}(3) \supsetneqq \cdots$$

Cyclic decomposition

Let $t, k, p \in \mathbb{N}$ and K be a set of size k.
Let Φ be a map from $\mathbb{Z}_p$ to Set_K.
Let $C_i := \Phi(i) \times \cdots \times \Phi(i + t - 1)$ for all $i \in \mathbb{Z}_p$.
We call Φ a cyclic decomposition of (K^t, t) with period $\mathrm{per}(\Phi) = p$ if $\{C_i\}_{i \in \mathbb{Z}_p}$ form a partition of K^t.



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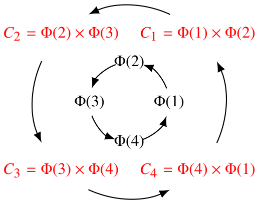
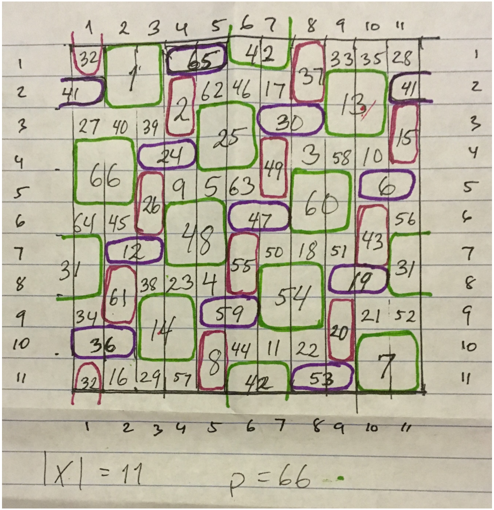


Theorem (Wu, Xu, Z., 2016+)

For any positive integer p, $p \in \mathcal{P}(t, k)$ if and only if there exists a cyclic decomposition Φ of (K^t, t) with $|K| = k$ and $\mathrm{per}(\Phi) = p$.

Examples of cyclic decomposition

- De Bruijn sequences
- An example from Bill Martin



A question

Question

Let t, k, p be three positive integers and let K be a set of size k. What is the number of different cyclic decompositions of (K^t, t) with period p?

A conjecture on expansion property

Conjecture (Wu, Xu, Z., 2016)

Let k be an integer larger than 3. Let A be a $k \times k$ primitive Boolean matrix with primitive exponent p. Let $\sigma(A)$ be the number of entries of A which equals 1.

- *If $\sigma(A) > k^2 - 4k + 7$, then*

$$\sigma(A) < \sigma(A^2) < \cdots < \sigma(A^p) = k^2.$$

- *If $\sigma(A) > k^2 - 5k + 10$, then*

$$\sigma(A) \leq \sigma(A^2) \leq \cdots \leq \sigma(A^p) = k^2.$$

THANK YOU!