Hamiltonian thickness and fault-tolerant spanning rooted path systems of graphs

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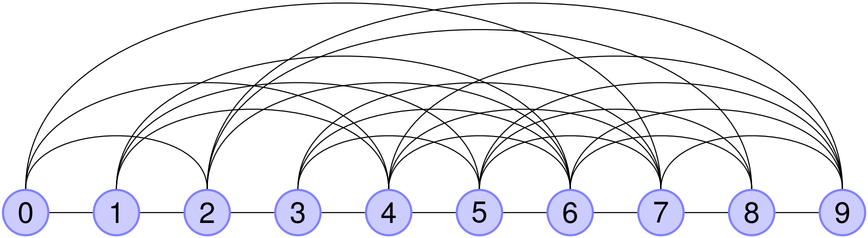
Joint with Yaokun Wu （吴耀琨）& Ziqing Xiang (向子卿)

Spanning connection pattern with fault-tolerance



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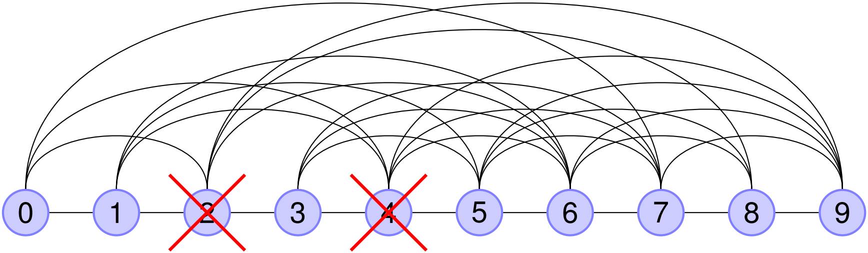

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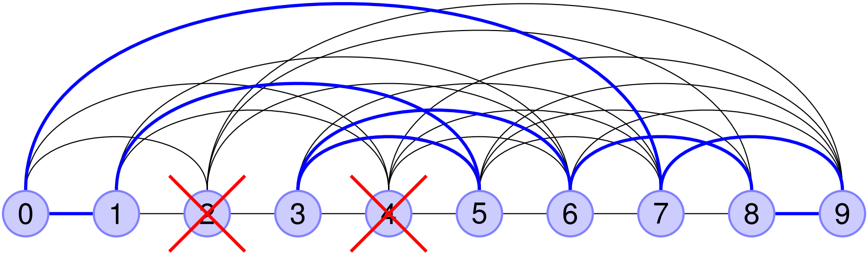

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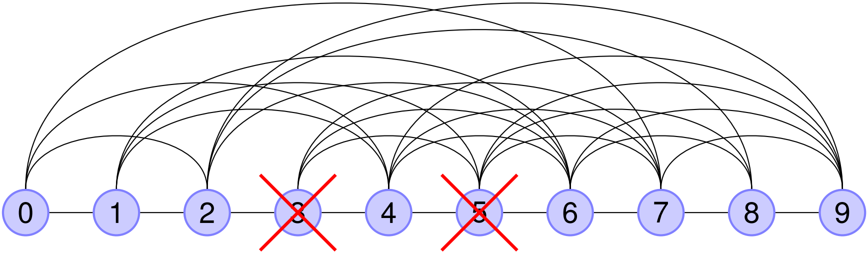

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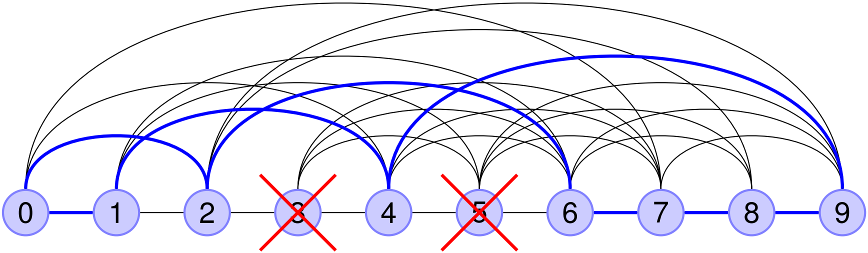

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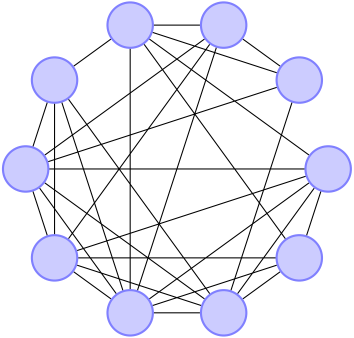

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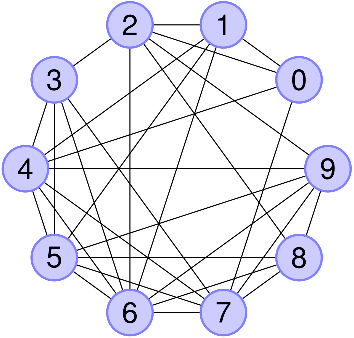

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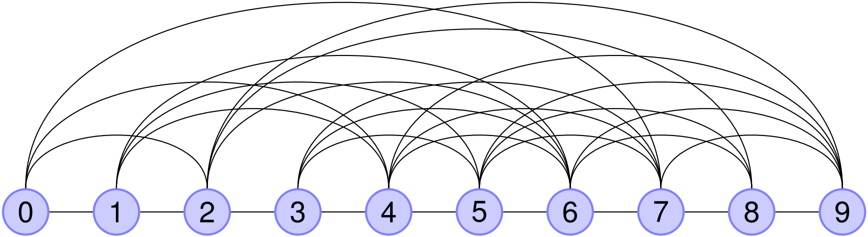
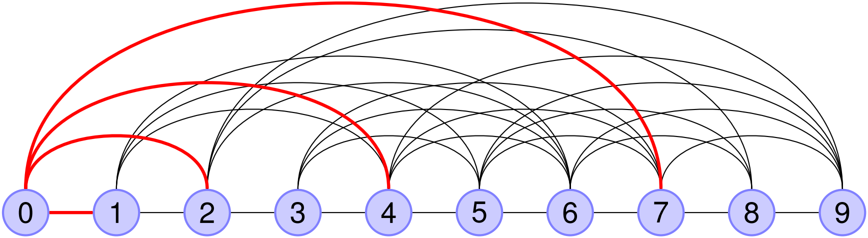


A 4-thick Hamiltonian vertex ordering.

Hamiltonian thickness



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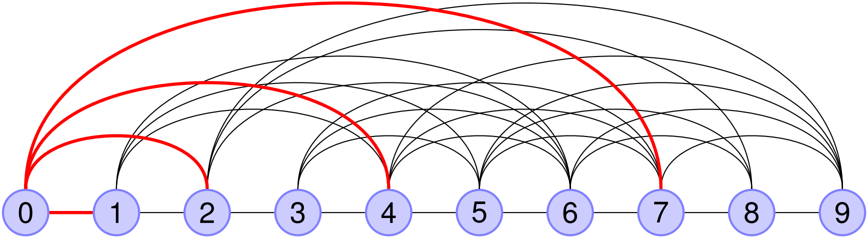

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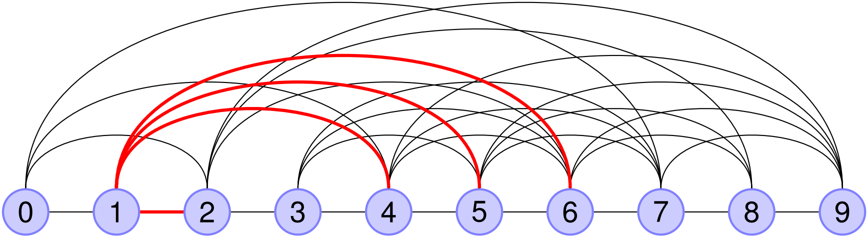

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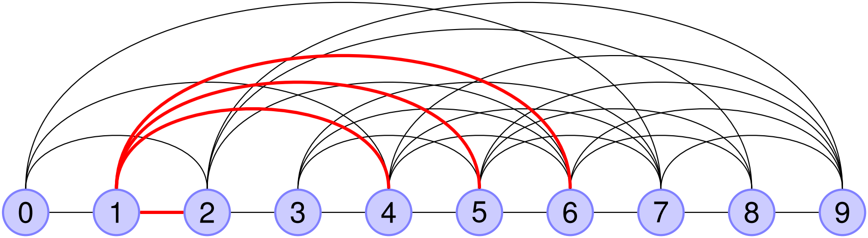

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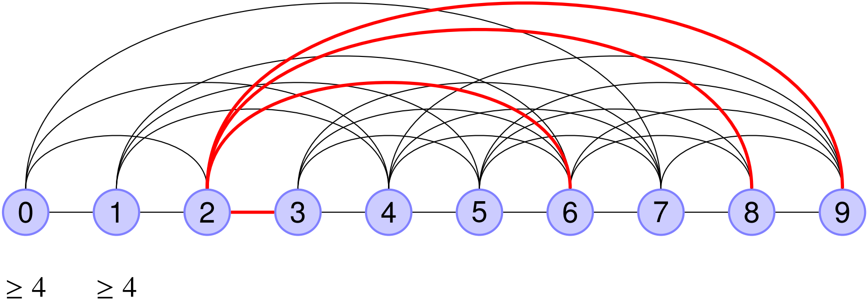

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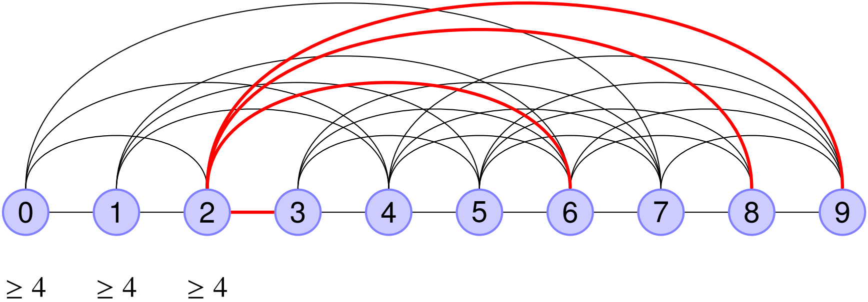

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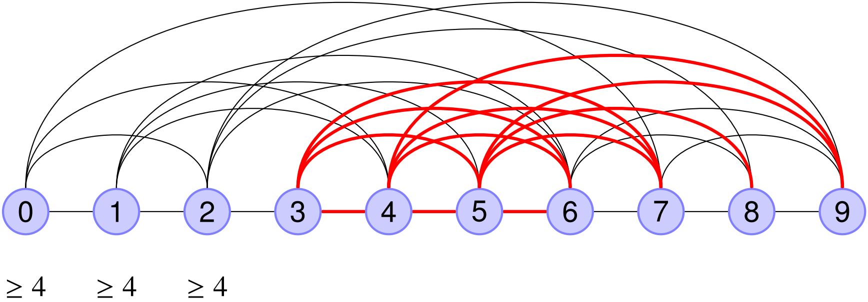

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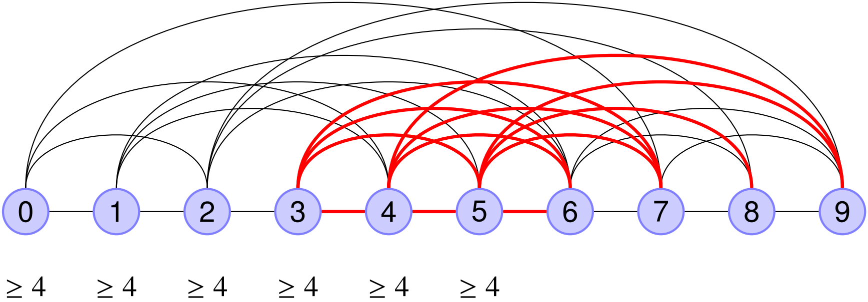

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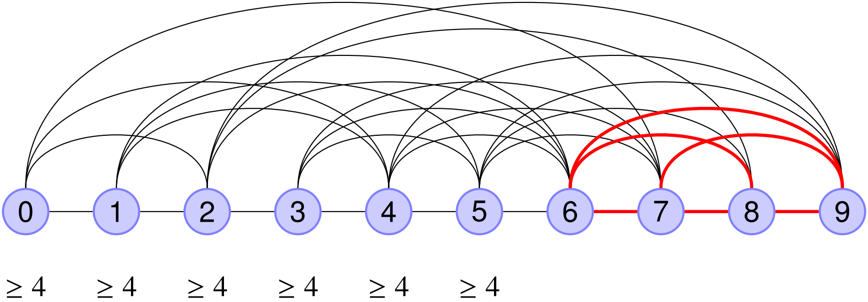

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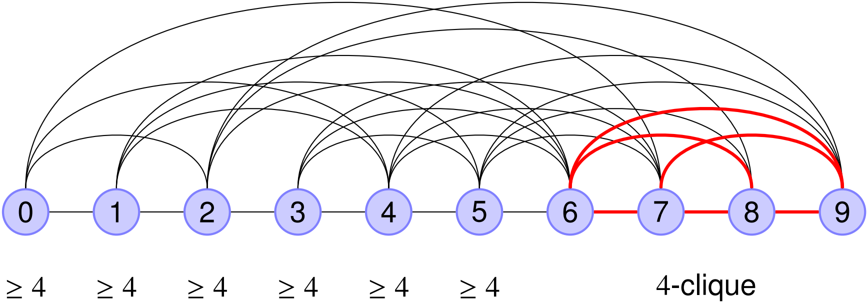

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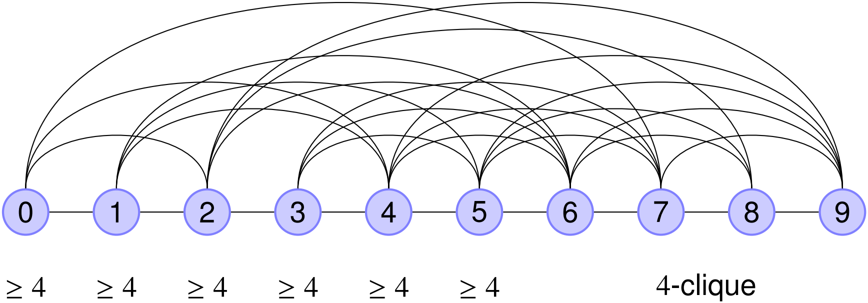

A 4-thick Hamiltonian vertex ordering.

Definition

A graph G is a k-thick Hamiltonian graph if G has a k-thick Hamiltonian vertex ordering.

Path system I

Let G be a graph and let H be a multigraph with $V_H \subseteq V_G$ and $|E_H| = m$.

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A path in G with identical endpoints is either a trivial path (of length zero) or a cycle (of length at least three). For a loop edge $vv \in E_H$, an vv-path in G should be understood as a cycle of length at least 3 but not any trivial path.

Path system II

A path system of G rooted at E_H is a set $Q = \{P_e : \ e \in E_H\}$ of m paths in G such that P_e is an e-path and every two distinct paths in the family intersect only at their possible common endpoints.

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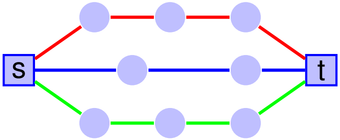
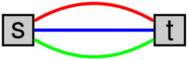


A spanning 3-rail rooted at E_H



H

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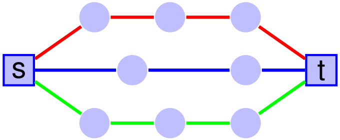

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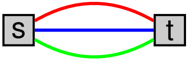

H

- spanning 1-rail ⇠⇢ Hamiltonian path
- spanning 2-rail ⇠⇢ Hamiltonian cycle

f-factor

Let H be a multigraph and $f : V_H \to \mathbb{N}$ be a map.

An f-factor of H is a multigraph F with

- $V_F = V_H$;
- $E_F \subseteq E_H$;
- $\deg_F(v) = f(v)$ for all $v \in V_F$.(Each loop contributes 2 degrees.)

The multigraph H is f-factor friendly if every g-factor of H can be extended into an f-factor of H by adding edges as long as $g \leq f$.

A path system of G rooted at (H, f)

Let G be a graph with $V_H \subseteq V_G$. A path system of G rooted at (H, f) is a path system of $G - f^{-1}(0)$ rooted at E_F for some f-factor F of H. A path system of G rooted at (H, f) is called spanning if every vertex from $V_G \setminus f^{-1}(0)$ appears in some path of the system.

Note that $f^{-1}(0)$ can be thought of as the set of faulty nodes in G.

Question and result

Assume that H is f-factor friendly.

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Which kind of (sparse) graphs have a (spanning) path system rooted at (H,f)?

We show that G has a (spanning) path system rooted at (H,f) whenever the Hamiltonian thickness of G is no less than some parameter determined by f.

Hamiltonian thickness vs. path system

Suppose that G has a vertex ordering $\pi_1, \ldots, \pi_n$ with Hamiltonian thickness at least k. Let H be an f-factor friendly multigraph.

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The graph G has a path system rooted at (H, f) provided $\pi_1 \in V_H$ and k is no less than

(i) $|f^{-1}(0)| + \sum_{v \in V_H} f(v) - \left\lceil \frac{1}{2} \min_{f(v) \neq 0} f(v) \right\rceil + 1$

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(H is bipartite with $\mathrm{m}_H(u, v) \geq \min\{f(u), f(v)\}$)

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If $\pi_1 \notin V_H$, the bounds for thickness should be increased by one.

Hamiltonian thickness vs. spanning path system

If $\pi_n \in V_H$ and $f(\pi_n) \geq 2$, the same bound applies for the existence of a spanning path system.

In remaining cases, we may need to increase the thickness bound by one.

Fault-tolerant spanning k-rail

Corollary

Let k and t be integers satisfying $k \geq t \geq 2$ and let G be a graph with a k-thick Hamiltonian vertex ordering $\pi_1, \ldots, \pi_n$. For every $C \subseteq \binom{V_G \setminus \{\pi_1, \pi_n\}}{\leq k-t}$, $G - C$ has a spanning t-rail between π_1 and π_n.

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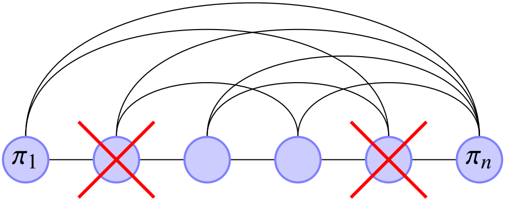


$t = 1, |C| = k - t$, no spanning t-rail between π_1 and π_n in $G - C$.

The greedy strategy, I

We start from a path family $\mathcal{P}$ which consists of $f(v)$ trivial paths at vertex v for all $v \in V_H$ as well as one trivial path at v for all $v \in V_G \setminus V_H$.

At each iteration, we find a suitable edge of G to connect two paths in $\mathcal{P}$ into one and update $\mathcal{P}$ accordingly. We always try to add the edge so that the path family obtained does not violate some obvious conditions for a path family to be further updated into a path system rooted at (H,f) (together with some other paths in $G - V_H$, if that path system is not required to be spanning).

A vertex $v \in V_G$ is finished in $\mathcal{P}$ if $v \in V_H$ and v is incident to $\deg_H(v)$ edges in $\mathcal{P}$ or $v \notin V_H$ and v is incident to 2 edges in $\mathcal{P}$. A vertex $v \in V_G$ is fresh in $\mathcal{P}$ if $v \notin V_H$ and v is incident to no edges in $\mathcal{P}$.

The greedy strategy, II

Let $\pi_1, \ldots, \pi_n$ be an ordering of V_G with high Hamiltonian thickness.

(i) Find the leftmost unfinished vertex in $\mathcal{P}$, say π_i. If π_i is fresh or $\pi_i \in V_H$, we choose $P \in \mathcal{P}$ as a trivial path at π_i. Else, we choose P as the unique nontrivial path in $\mathcal{P}$ that contains π_i as an endpoint.

(ii) We try to find another unfinished vertex $\pi_j \in N_G(\pi_i)$ and a path $Q \in \mathcal{P}$ containing π_j and connect P and Q by adding edge $\pi_i\pi_j$. We should guarantee that the updated path family is still "allowed".

(iii) If π_{i+1} is fresh, then we should choose $j = i + 1$.

The greedy strategy, III

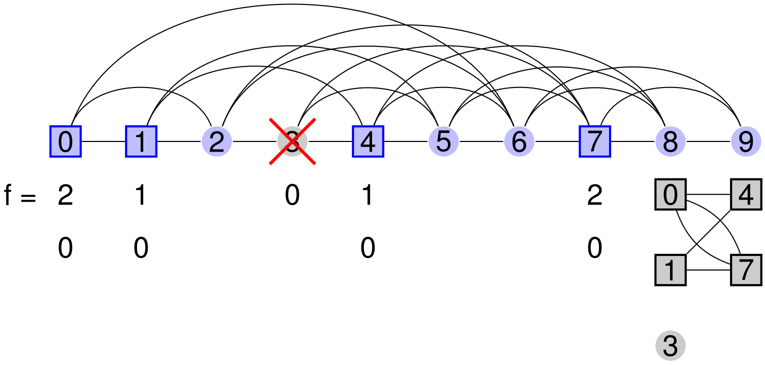
Suppose we want to find a spanning path system rooted at (H, f). We follow two more strategies.

(i) Whenever possible, we should first try to choose π_j so that the updated path family $\mathcal{P}$ does not contain a subdivision of any f-factor of H.

(ii) Suppose $\pi_n \in V_H$ and $f(\pi_n) \geq 2$. If π_n is not on a nontrivial path in the path family $\mathcal{P}$ yet, then we choose $j = n$ only when there is no other choice.

Finding spanning path system

$\mathcal{P} = \{0, 0, 1, 2, 4, 5, 6, 7, 7, 8, 9\}$
$\mathcal{Q} = \{\}$
$P = ?$
$Q = ?$



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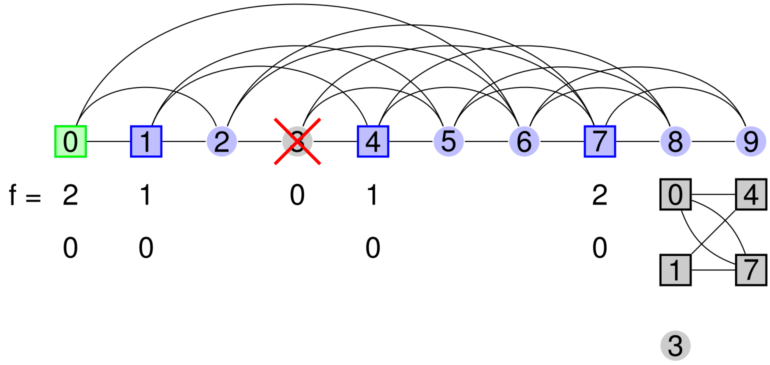

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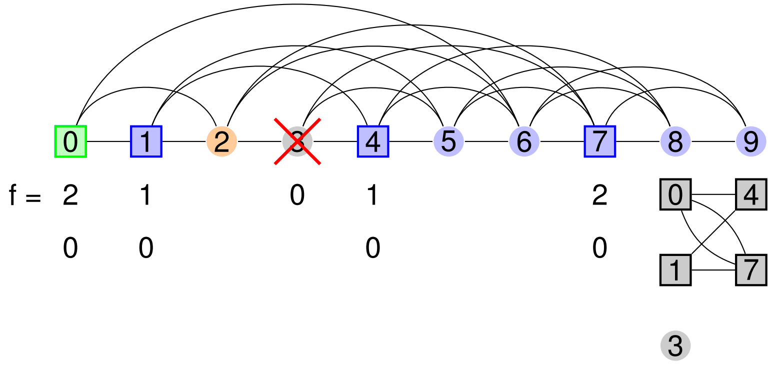
$P = 0$

$Q = 2$



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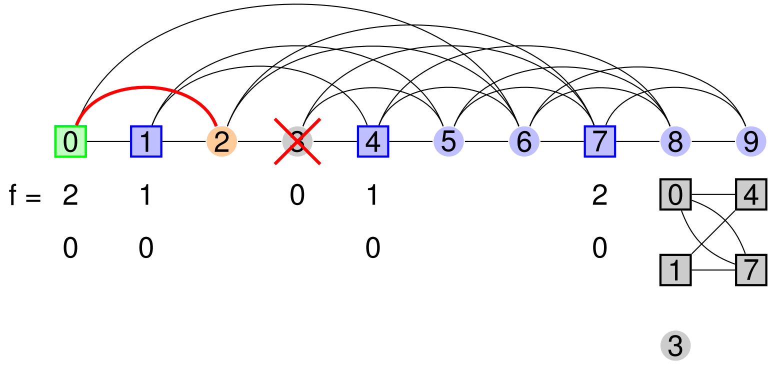

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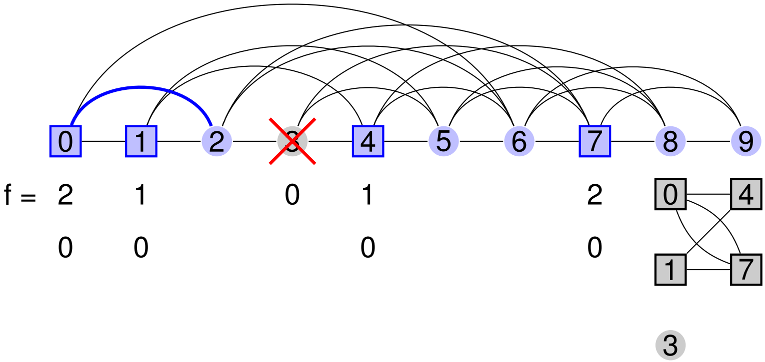

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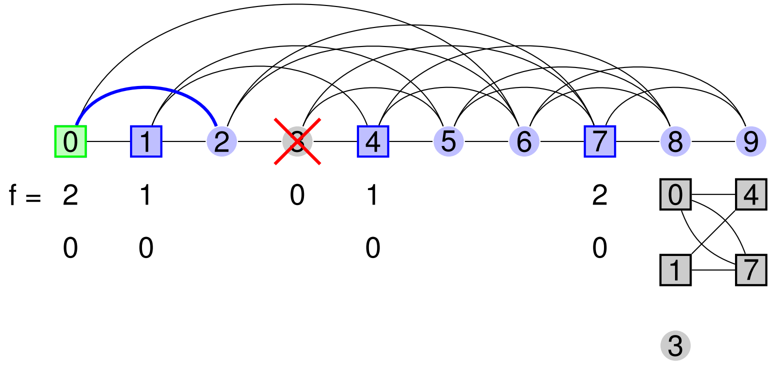


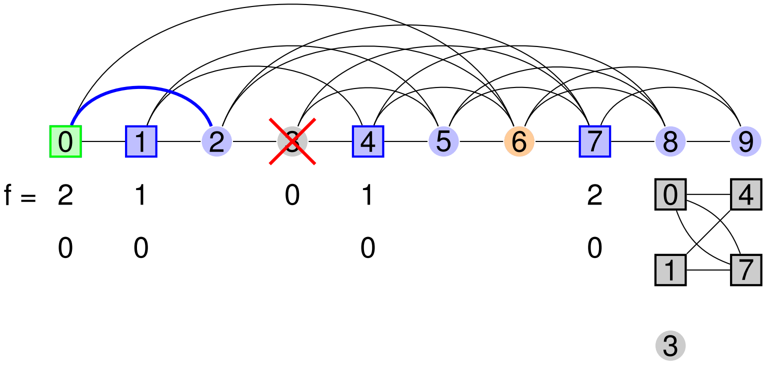
Finding spanning path system

$\mathcal{P}$ = {0-2, 0, 1, 4, 5, 6, 7, 7, 8, 9}

$\mathcal{Q}$ = {}

P =0

Q =6



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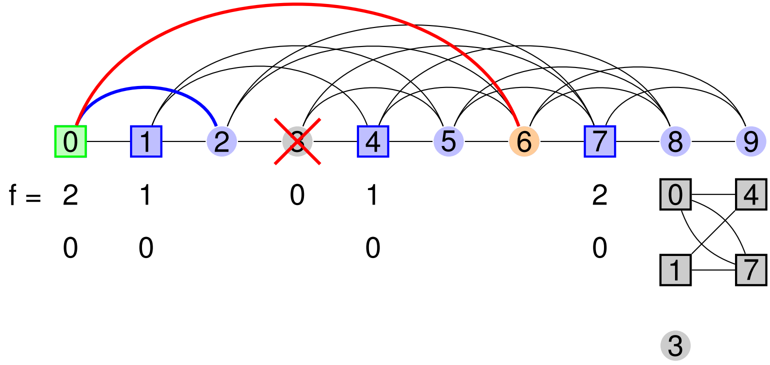

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$P = ?$

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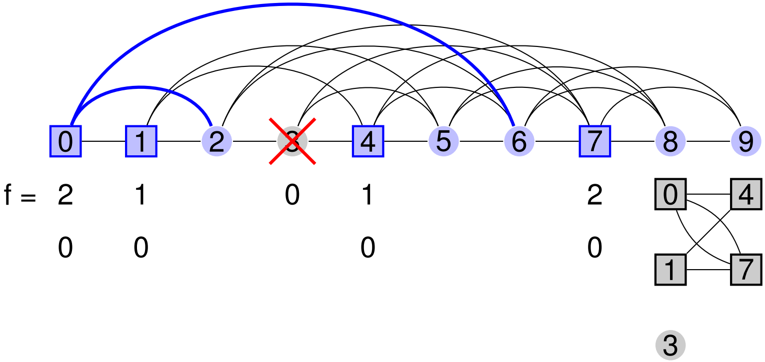


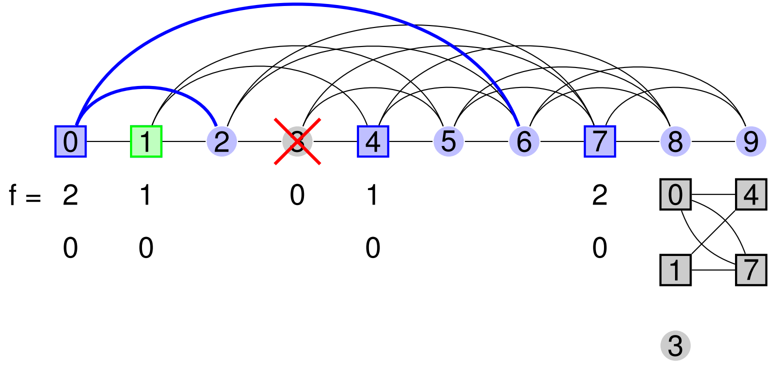
Finding spanning path system

$\mathcal{P} = \{0\text{-}2, 0\text{-}6, 1, 4, 5, 7, 7, 8, 9\}$

$\mathcal{Q} = \{\}$

$P = 1$

$Q = ?$



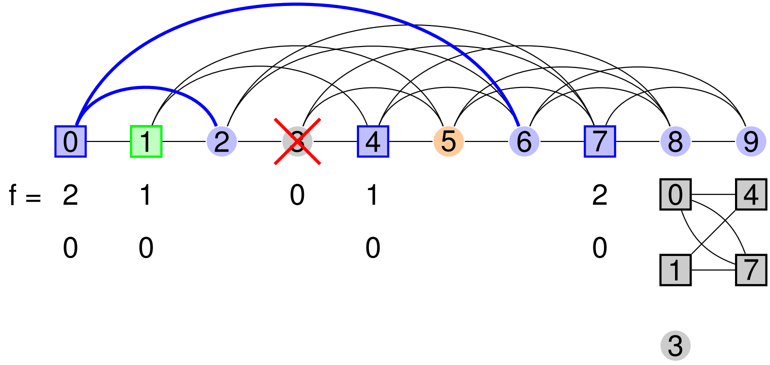
Finding spanning path system

$\mathcal{P} = \{0\text{-}2, 0\text{-}6, 1, 4, 5, 7, 7, 8, 9\}$

$\mathcal{Q} = \{\}$

$P = 1$

$Q = 5$



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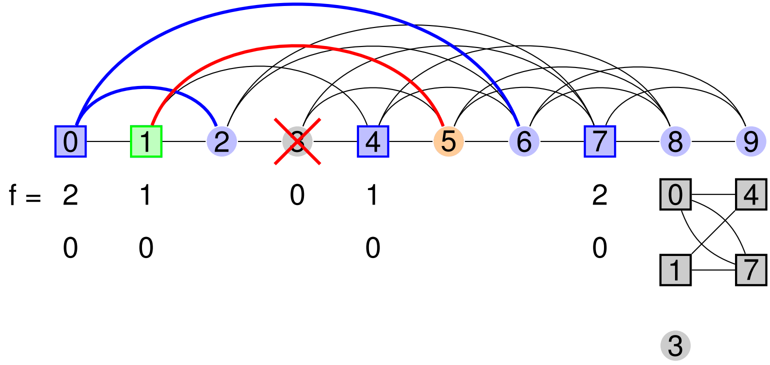
$\mathcal{Q} = \{\}$

$P = 1$

$Q = 5$



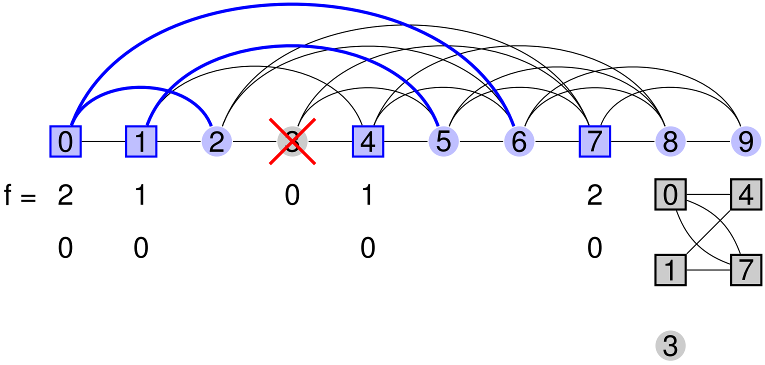
Finding spanning path system

$\mathcal{P} = \{0\text{-}2, 0\text{-}6, 1\text{-}5, 4, 7, 7, 8, 9\}$

$\mathcal{Q} = \{\}$

$P = ?$

$Q = ?$



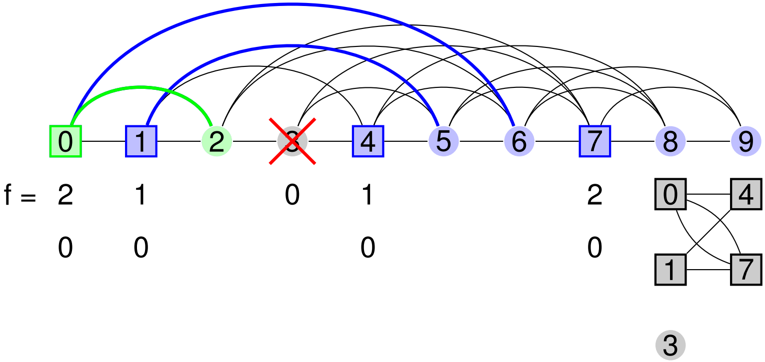
Finding spanning path system

$\mathcal{P}$ = {0-2, 0-6, 1-5, 4, 7, 7, 8, 9}

Q = {}

P =0-2

Q =?



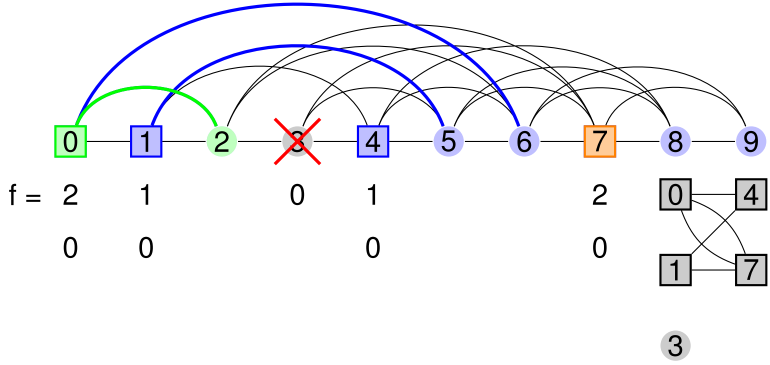
Finding spanning path system

$\mathcal{P} = \{0\text{-}2, 0\text{-}6, 1\text{-}5, 4, 7, 7, 8, 9\}$

$\mathcal{Q} = \{\}$

P =0-2

Q =7



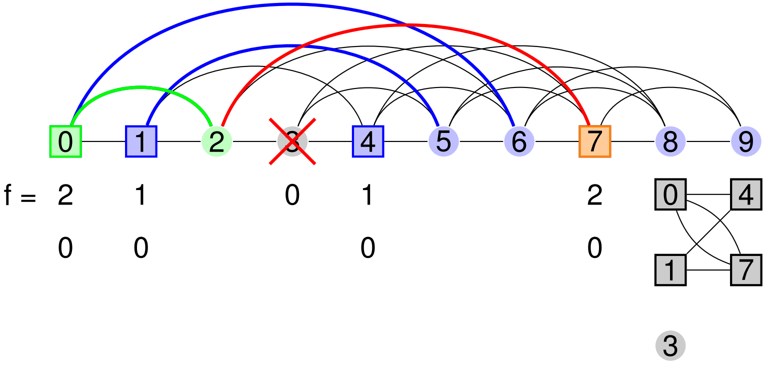
Finding spanning path system

$\mathcal{P} = \{0\text{-}2, 0\text{-}6, 1\text{-}5, 4, 7, 7, 8, 9\}$

$\mathcal{Q} = \{\}$

$P = 0\text{-}2$

$Q = 7$



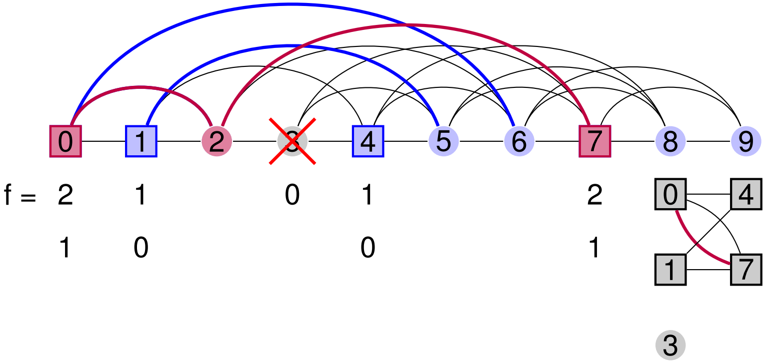
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4, 7, 8, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

$P = ?$

$Q = ?$



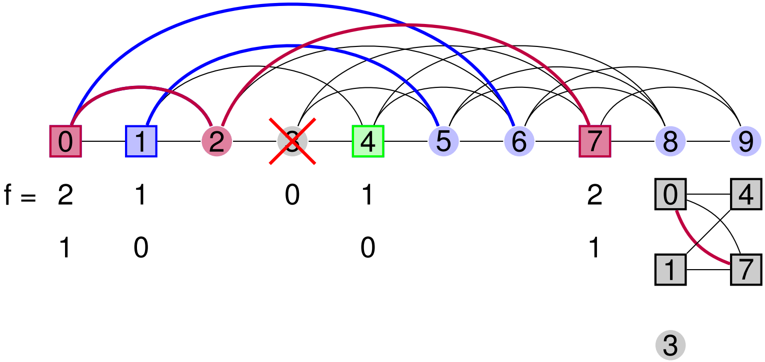
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4, 7, 8, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

$P = 4$

$Q = ?$



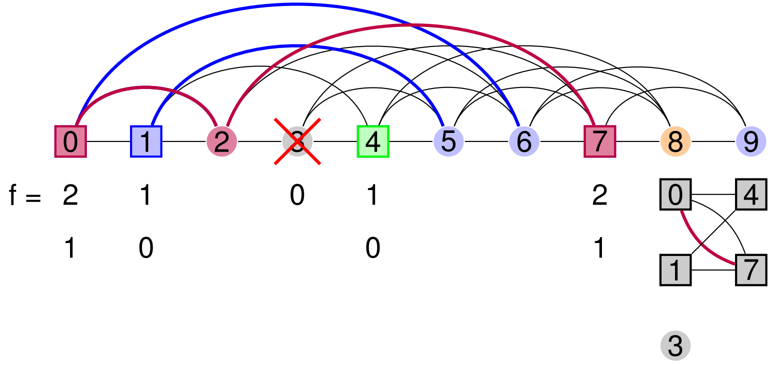
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4, 7, 8, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

$P = 4$

$Q = 8$



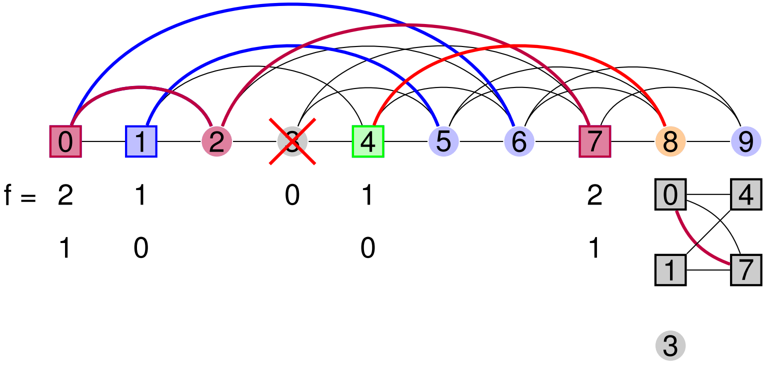
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4, 7, 8, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

$P = 4$

$Q = 8$



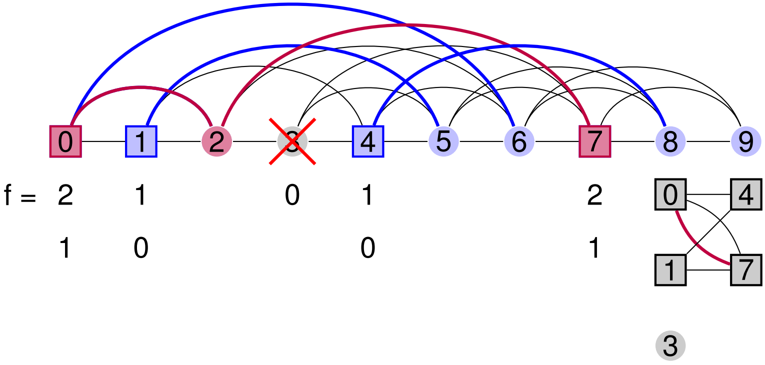
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4\text{-}8, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

$P = ?$

$Q = ?$



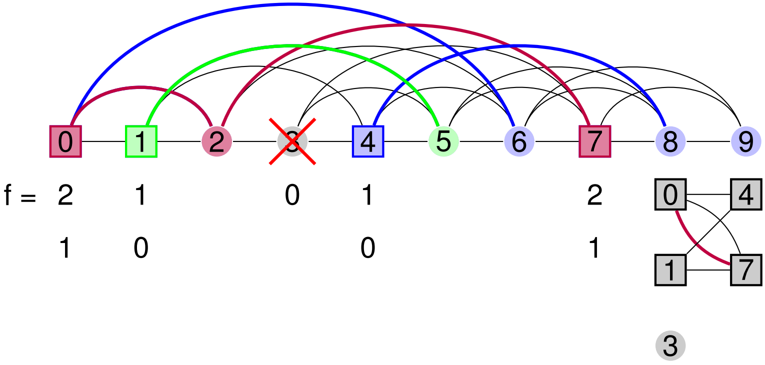
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4\text{-}8, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

$P =$1-5

$Q =$?



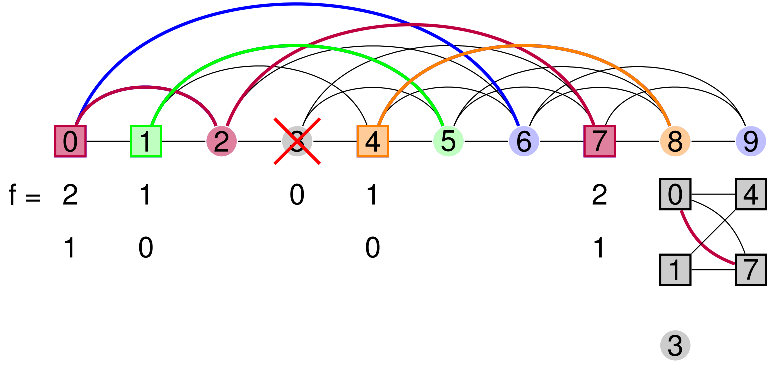
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 1\text{-}5, 4\text{-}8, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7\}$

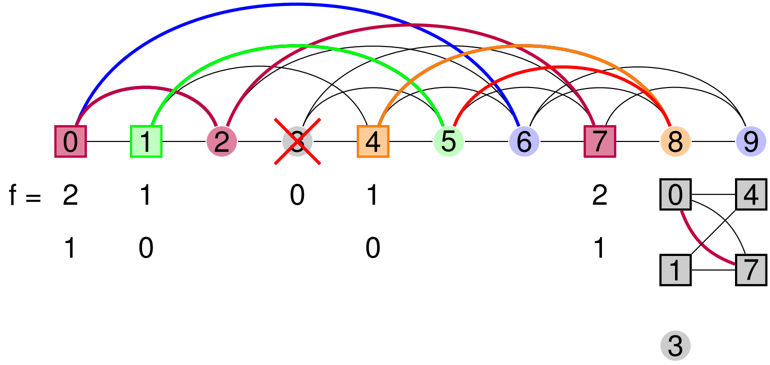
$P = 1\text{-}5$

$Q = 4\text{-}8$



Finding spanning path system

$\mathcal{P}$ = {0-6, 1-5, 4-8, 7, 9}
$\mathcal{Q}$ = {0-2-7}
P =1-5
Q =4-8



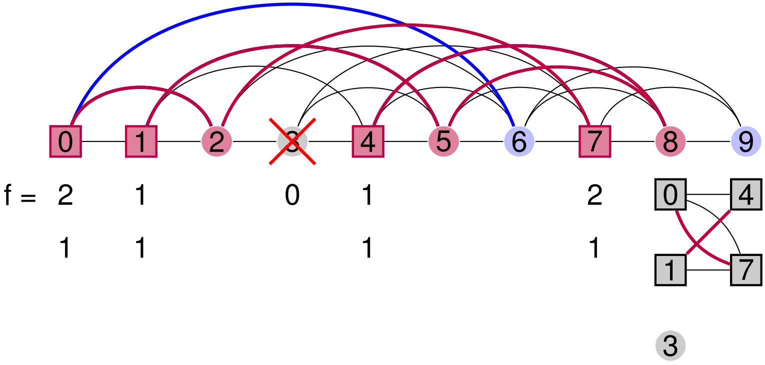
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

$P =?$

$Q =?$



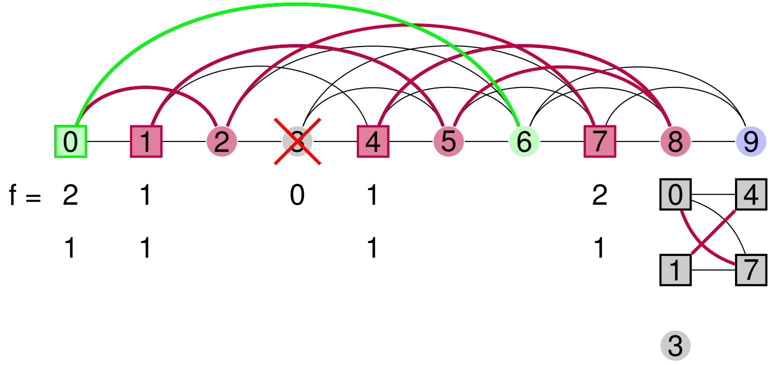
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

$P = 0\text{-}6$

$Q = ?$



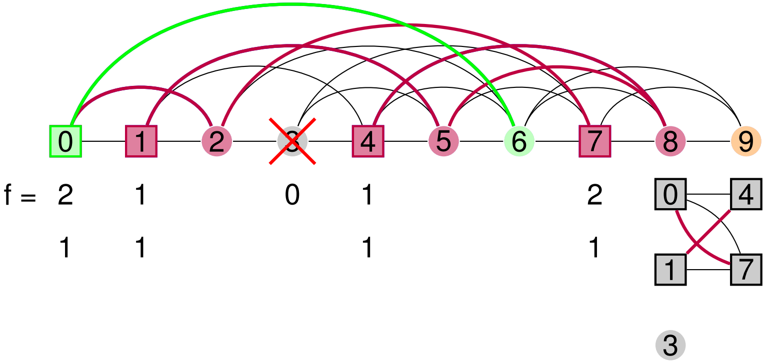
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

$P = 0\text{-}6$

$Q = 9$



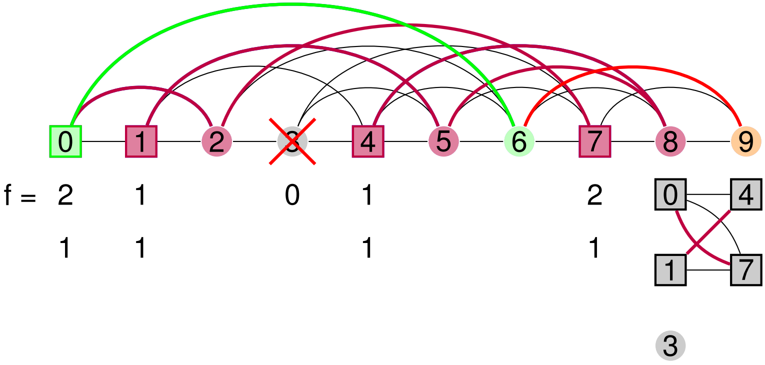
Finding spanning path system

$\mathcal{P} = \{0\text{-}6, 7, 9\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

$P = 0\text{-}6$

$Q = 9$



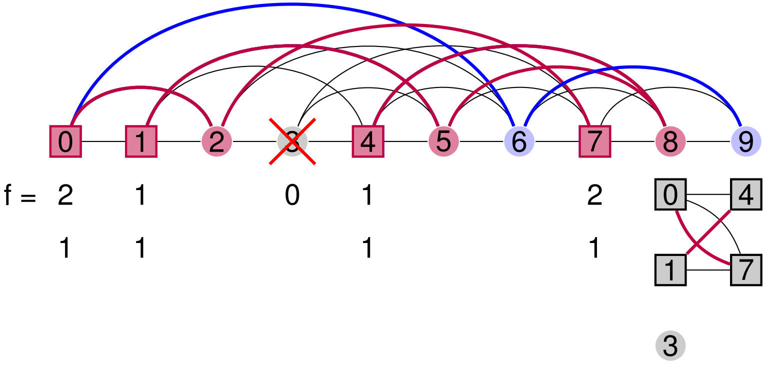
Finding spanning path system

$\mathcal{P} = \{0\text{-}6\text{-}9, 7\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

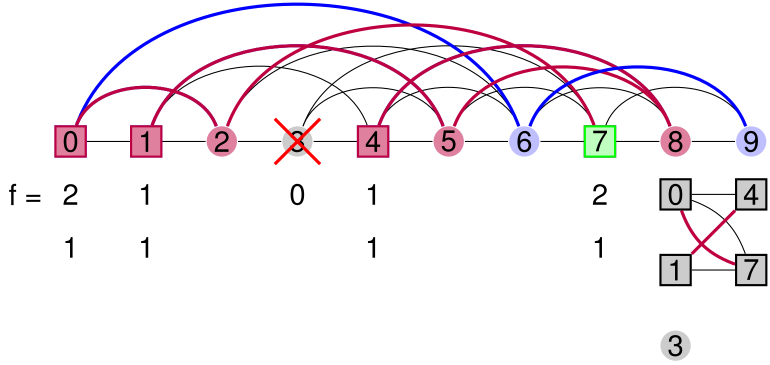
$P = ?$

$Q = ?$



Finding spanning path system

$\mathcal{P} = \{0\text{-}6\text{-}9, 7\}$
$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$
$P = 7$
$Q = ?$



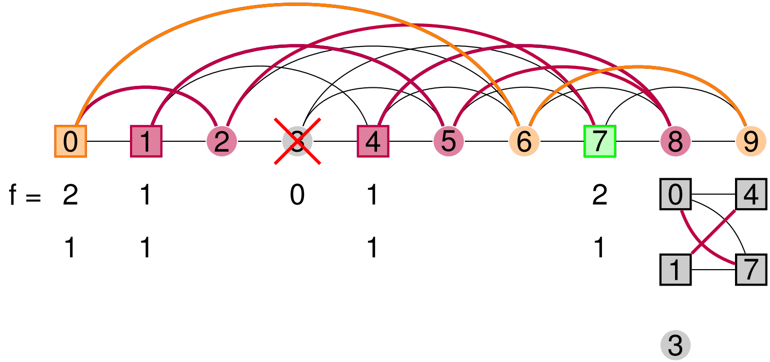
Finding spanning path system

$\mathcal{P} = \{0\text{-}6\text{-}9, 7\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

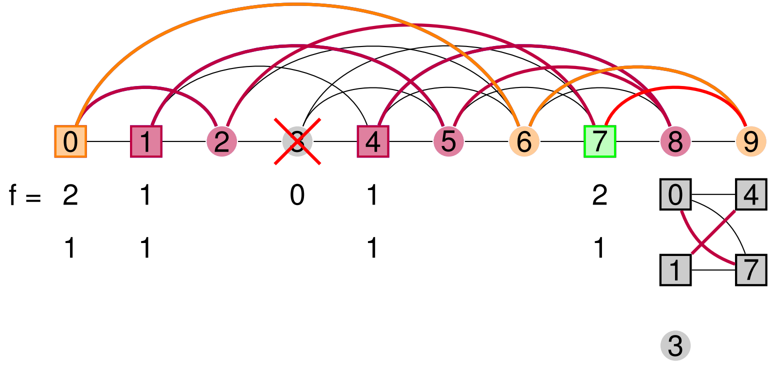
$P = 7$

$Q = 0\text{-}6\text{-}9$



Finding spanning path system

$\mathcal{P}$ = {0-6-9, 7}
$\mathcal{Q}$ = {0-2-7, 1-5-8-4}
P =7
Q =0-6-9



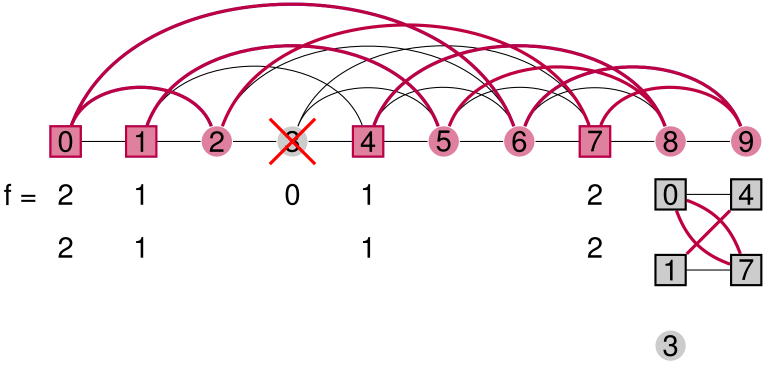
Finding spanning path system

$\mathcal{P} = \{\}$

$\mathcal{Q} = \{0\text{-}2\text{-}7, 1\text{-}5\text{-}8\text{-}4, 0\text{-}6\text{-}9\text{-}7\}$

$P = ?$

$Q = ?$



A monotonicity lemma

Assume that the algorithm has been running successfully to produce the path family $\mathcal{P}_t$ with $|E_{\mathcal{P}_t}| = t$ for some nonnegative integer t.

Let Σ_t be the set of finished vertices for $\mathcal{P}_t$.
Denote by Ω_t the family of paths containing no fresh vertices in the path family $\mathcal{P}_t$.

If $\pi_1 \in V_H \setminus f^{-1}(0)$, then
$|\Sigma_t| + |\Omega_t| \le |\Sigma_{t-1}| + |\Omega_{t-1}| \le \cdots \le |\Sigma_0| + |\Omega_0| = |f^{-1}(0)| + \sum_{v \in V_H} f(v)$.

Monotonicity of $|\Sigma_t| + |\Omega_t|$

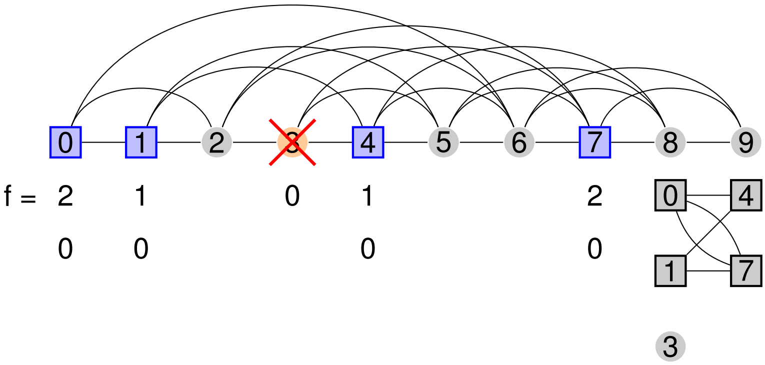
$t = 0$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{3\}$

$\Omega_t = \{0, 0, 1, 4, 7, 7\}$

$|\Sigma_t| + |\Omega_t| = 7$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

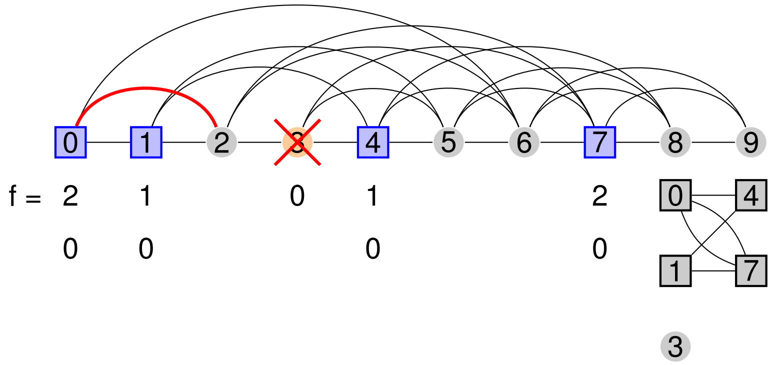
$t = 1$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{3\}$

$\Omega_t = \{0, 0\text{-}2, 1, 4, 7, 7\}$

$|\Sigma_t| + |\Omega_t| = 7$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

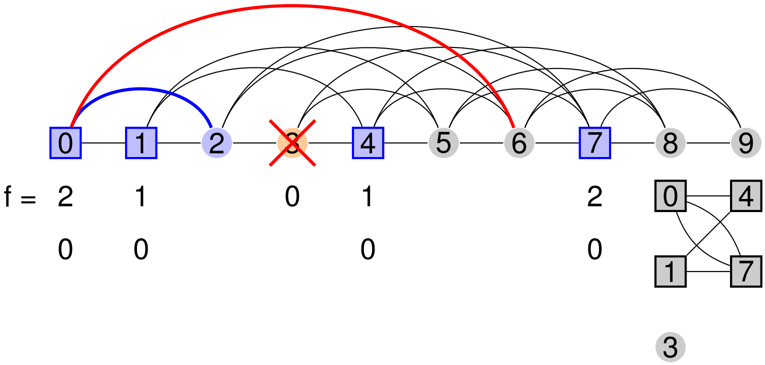
$t = 2$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{3\}$

$\Omega_t = \{0\text{-}2, 0\text{-}6, 1, 4, 7, 7\}$

$|\Sigma_t| + |\Omega_t| = 7$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

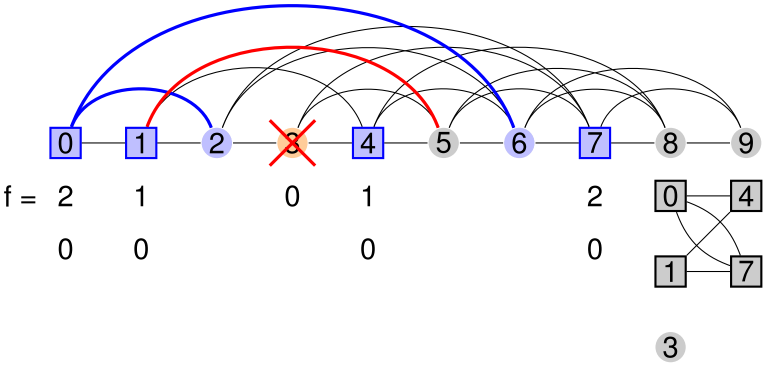
$t = 3$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{3\}$

$\Omega_t = \{0\text{-}2, 0\text{-}6, 1\text{-}5, 4, 7, 7\}$

$|\Sigma_t| + |\Omega_t| = 7$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

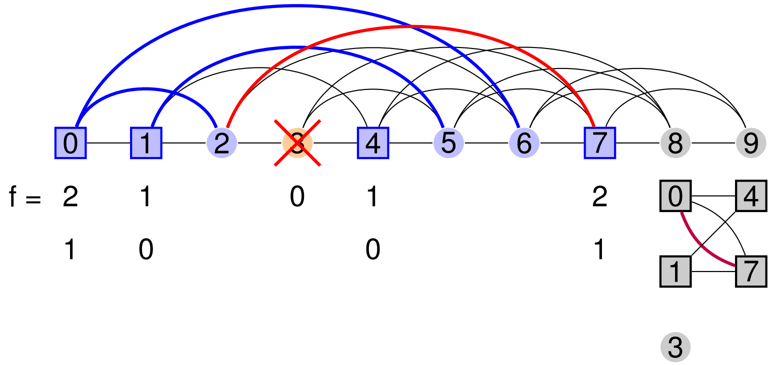
$t = 4$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{3\}$

$\Omega_t = \{0\text{-}2\text{-}7, 0\text{-}6, 1\text{-}5, 4, 7\}$

$|\Sigma_t| + |\Omega_t| = 6$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

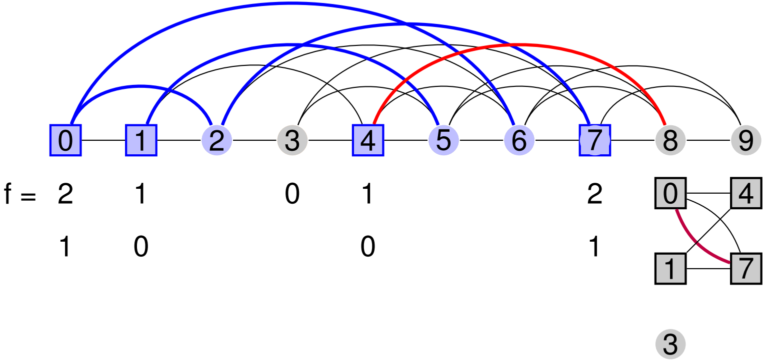
$t = 5$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{\}$

$\Omega_t = \{0\text{-}2\text{-}7, 0\text{-}6, 1\text{-}5, 4\text{-}8, 7\}$

$|\Sigma_t| + |\Omega_t| = 5$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

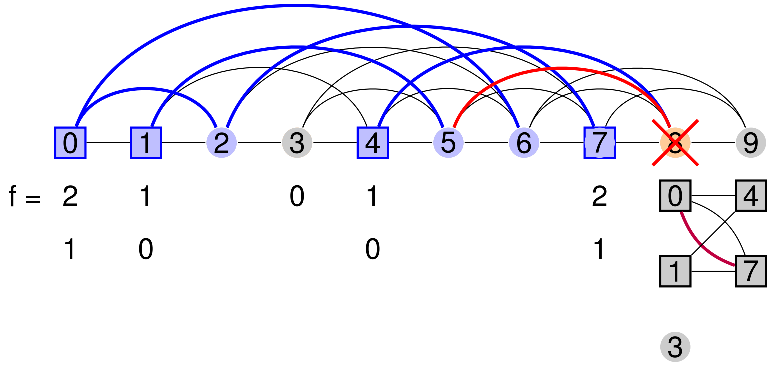
$t = 6$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{8\}$

$\Omega_t = \{0\text{-}2\text{-}7, 0\text{-}6, 1\text{-}5\text{-}8\text{-}4, 7\}$

$|\Sigma_t| + |\Omega_t| = 5$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

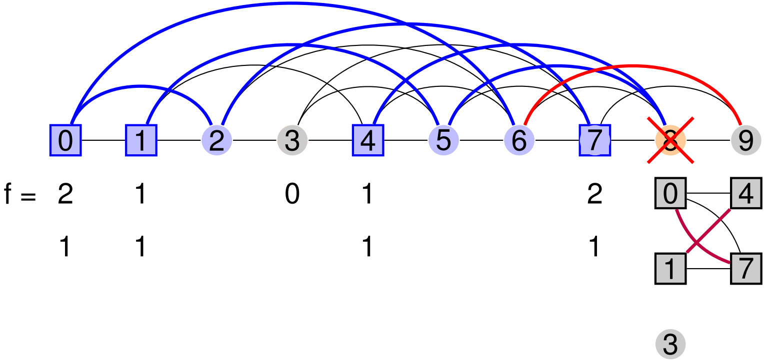
$t = 7$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{8\}$

$\Omega_t = \{0\text{-}2\text{-}7, 0\text{-}6\text{-}9, 1\text{-}5\text{-}8\text{-}4, 7\}$

$|\Sigma_t| + |\Omega_t| = 5$



Monotonicity of $|\Sigma_t| + |\Omega_t|$

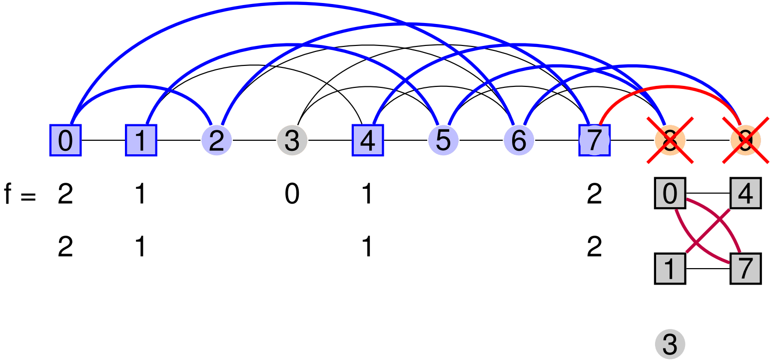
$t = 8$

$E = (0\text{-}2, 0\text{-}6, 1\text{-}5, 2\text{-}7, 4\text{-}8, 5\text{-}8, 6\text{-}9, 7\text{-}9)$

$\Sigma_t = \{8, 9\}$

$\Omega_t = \{0\text{-}2\text{-}7, 0\text{-}6\text{-}9\text{-}7, 1\text{-}5\text{-}8\text{-}4\}$

$|\Sigma_t| + |\Omega_t| = 5$



The end

Thank you

Tolerating edge failures

We can increase the Hamiltonian thickness bound by one to deal with every edge failure.