PRIMITIVITY AND HURWITZ PRIMITIVITY OF NONNEGATIVE MATRIX TUPLES: A UNIFIED APPROACH*

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Abstract. For an m-tuple of nonnegative $n{\times}n$ matrices $(A_1, \ldots, A_m)$, primitivity/Hurwitz primitivity means the existence of a positive product/Hurwitz product respectively (all products are with repetitions permitted). The Hurwitz product with a Parikh vector $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$ is the sum of all products with α_i multipliers A_i, $i = 1, \ldots, m$. Ergodicity/Hurwitz ergodicity means the existence of the corresponding product with a positive row.

We give a unified proof for the Protasov-Vonyov characterization (2012) of primitive tuples of matrices without zero rows and columns and for the Protasov characterization (2013) of Hurwitz primitive tuples of matrices without zero rows. By establishing a connection with synchronizing automata, we, under the aforementioned conditions, find an $O(n^2m)$-time algorithm to decide primitivity and an $O(n^3m^2)$-time algorithm to construct a Hurwitz primitive vector α of weight $\sum_{i=1}^m \alpha_i = O(n^3)$. We also report results on ergodic and Hurwitz ergodic matrix tuples.

Key words. automaton, Černý function, ergodic exponent, Hamiltonian walk, primitive exponent, stable relation.

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1. Primitivity and Hurwitz primitivity.

Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers and let $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ be the set of n by n nonnegative real matrices. Various dynamical behaviors for homogeneous, inhomogeneous and high dimensional Markov chains lead to the study of various nonnegative matrix classes and their Boolean counterparts [25, 47, 49]. A matrix $A \in \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ is *primitive* if some positive power of A is a positive matrix; a matrix $A \in \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ is *ergodic* if some positive power of A has a positive column. A finite Markov chain (resp., irreducible Markov chain) has a unique stationary distribution if and only if its transition matrix is ergodic (resp., primitive) [27, 36]. Note that an ergodic matrix is also named as column-primitive and a stochastic matrix is ergodic if and only if it is stochastic indecomposable aperiodic [11, Proposition 1][12, Proposition 1]. For an ergodic (resp., primitive) matrix A, it is of interest to estimate the minimum positive integer k such that A^k has a positive column (resp., is a positive matrix). There are several possibilities to generalize this concept from homogeneous chains to inhomogeneous chains, namely from a matrix to a set of matrices. This paper is about two of them, primitivity/ergodicity and Hurwitz primitivity/ergodicity. This is part of the study of the general reachability problem, for which we refer to [44, 60] for a glimpse of a broader scope of research.

We write $\mathbb{Z}_{\geq 0}$ for the set of nonnegative integers and write $\mathbb{N}$ for the set of positive integers. For any real number x, we use $[x]$ to denote the set $\{i \in \mathbb{N} : \ i \leq x\}$. Let

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X be a set. A *word of length s over X* is a sequence of elements from X of length s, say $\alpha_1 \cdots \alpha_s$ where $\alpha_1, \dots, \alpha_s \in X$. Let $\alpha = \alpha_1 \cdots \alpha_s$ and $\beta = \beta_1 \cdots \beta_t$ be two words over X. We write $\alpha\beta$ for the word $\alpha_1 \cdots \alpha_s \beta_1 \cdots \beta_t$. For each $x \in X$, we denote the number of occurrences of x in the word α by $|\alpha|_x$, that is $|\alpha|_x = |\{i \in [s] : \alpha_i = x\}|$. The *Parikh vector* of α, dubbed by $\Psi(\alpha)$, is defined as the vector in $\mathbb{Z}_{\geq 0}^X$ such that $\Psi(\alpha)(x) = |\alpha|_x$ for all $x \in X$ [48]. Note that $\Psi(\alpha)$ is known as the content [22, p. 3] or type [39, p. 52] of α in the study of Young tableaux, and is called the color vector of α by some authors [42, 43]. When $X = [m]$, we often write $\mathbb{Z}_{\geq 0}^X$ as $\mathbb{Z}_{\geq 0}^m$. For any $\tau \in \mathbb{Z}_{\geq 0}^m$, we adopt the notation $|\tau|$ for $\sum_{i=1}^m \tau(i)$ and call it the *weight* of τ.

Let $\mathcal{A} = (A_1, \dots, A_m)$ be an m-tuple of $n \times n$ nonnegative matrices, namely $\mathcal{A}$ is a map from $[m]$ to $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ that sends $i \in [m]$ to A_i. For each word $\alpha = \alpha_1 \cdots \alpha_s \in [m]^s$, we denote by $\mathcal{A}_\alpha$ the matrix $A_{\alpha_1} \cdots A_{\alpha_s}$ and call it a *product over $\mathcal{A}$ of length s*. For any $\tau \in \mathbb{Z}_{\geq 0}^m$, let $\mathcal{A}^\tau$ denote the matrix $\sum_{\Psi(\alpha)=\tau} \mathcal{A}_\alpha$. We name $\mathcal{A}^\tau$ a *Hurwitz product of $\mathcal{A}$ of* length $|\tau|$. A word α over $[m]$ is a *primitive word* for $\mathcal{A}$ provided $\mathcal{A}_\alpha > 0$, and it is an *ergodic word* for $\mathcal{A}$ provided $\mathcal{A}_\alpha$ has a positive column; a vector $\tau \in \mathbb{Z}_{\geq 0}^m$ with positive weight is called a *Hurwitz primitive vector* of $\mathcal{A}$ if $\mathcal{A}^\tau > 0$, and it is called a *Hurwitz ergodic vector* of $\mathcal{A}$ if $\mathcal{A}^\tau$ has a positive column. We call $\mathcal{A}$ *primitive* [45] (resp., *ergodic* [45], *Hurwitz primitive* [15], *Hurwitz ergodic* [23]) if it has a primitive word (resp., ergodic word, Hurwitz primitive vector, Hurwitz ergodic vector). The *primitive exponent* and *ergodic exponent* of $\mathcal{A}$, denoted by $\mathsf{p}(\mathcal{A})$ and $\mathsf{e}(\mathcal{A})$, respectively, are the minimum length of a primitive word and an ergodic word of $\mathcal{A}$; the *Hurwitz primitive exponent* and the *Hurwitz ergodic exponent* of $\mathcal{A}$, denoted by $\mathsf{hp}(\mathcal{A})$ and $\mathsf{he}(\mathcal{A})$, respectively, are the minimum weight of a Hurwitz primitive vector and a Hurwitz ergodic vector of $\mathcal{A}$. We use the convention that the exponent is ∞ when the corresponding word/vector does not exist. We will denote the largest finite value of $\mathsf{p}(\mathcal{A})$, $\mathsf{hp}(\mathcal{A})$, $\mathsf{e}(\mathcal{A})$, $\mathsf{he}(\mathcal{A})$ by $\mathsf{p}(n,m)$, $\mathsf{hp}(n,m)$, $\mathsf{e}(n,m)$, $\mathsf{he}(n,m)$, respectively, where $\mathcal{A}$ runs through all m-tuples of $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$.

The concept of primitivity for nonnegative matrix families has appeared in the study of Lyapunov exponents of random matrix products [40], stochastic control, refinement equations [56], consensus problems, mathematical ecology, scrambling matrices and Boolean networks [5, 21]. Hurwitz primitivity for nonnegative matrix families has background in multivariate Markov chains [15, 16, 17]. Hurwitz ergodicity and some related concepts are closely related to synchronizing problems for automata [23, 43, 44].

Given any m-tuple $\mathcal{A}$ over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$, one would like to determine if it is primitive (resp., Hurwitz primitive, ergodic, Hurwitz ergodic); moreover, one may like to find a product (resp., Hurwitz product) of $\mathcal{A}$ which is positive or has a positive column. These problems are closely related to the problems of bounding the corresponding exponents.

Martyugin finds that it is PSPACE-complete to decide whether a given size-two matrix set is ergodic [34, Proposition 2]. This further calls forth the result of Gerencsér, Gusev and Jungers that the problem of deciding whether a given set of two matrices is primitive is also PSPACE-complete [21, Theorem 6]. As with how complex it is to decide Hurwitz primitivity or Hurwitz ergodicity for general matrix sets, there seems to be no result at all, to the best of our knowledge.

For each $n \in \mathbb{N}$, let $\mathsf{p}(n) = \max_{m \in \mathbb{N}} \mathsf{p}(n,m)$ and $\mathsf{e}(n) = \max_{m \in \mathbb{N}} \mathsf{e}(n,m)$. Rystsov [46, Theorem 2 and Eq. (6)] proves that $\lim_{n\to\infty} \frac{\log \mathsf{e}(n)}{n} = \frac{\log 3}{3}$. Gerencsér, Gusev and Jungers find that $\mathsf{p}(n) = \Theta(\mathsf{e}(n))$ [21, Theorem 2], which combined with the above result of Rystsov then leads to $\lim_{n\to\infty} \frac{\log \mathsf{p}(n)}{n} = \frac{\log 3}{3}$ [21, Theorem 3]. For each fixed $m \in \mathbb{N}$, it is discovered by Olesky, Shader, and Van den Driessche that $\mathsf{hp}(n,m) = \Theta(n^{m+1})$ [37, Theorem 7]. Take a positive integer n. A classical result of Wielandt [59] [60, Corollary 1.4] claims that $\mathsf{p}(n,1) = \mathsf{hp}(n,1) = 1 + (n-1)^2$. Based upon [11, Corollary 1] or [60, Lemma 2.1], it is not hard to check that $\mathsf{e}(n,1) = \mathsf{he}(n,1) = 1 + (n-2)(n-1)$; see Theorem 3.4.

2. Nonnegative matrices without zero rows/columns. The set of nonnegative matrices that has no zero rows is denoted by NZ_1 and the set of nonnegative matrices that has no zero rows and no zero columns is denoted by NZ_2. For every positive integer n, we use $\mathsf{NZ}_1(n)$ and $\mathsf{NZ}_2(n)$ as a shorthand for $\mathsf{Mat}_n(\mathbb{R}_{\geq 0}) \cap \mathsf{NZ}_1$ and $\mathsf{Mat}_n(\mathbb{R}_{\geq 0}) \cap \mathsf{NZ}_2$, respectively. We shall focus our attention on NZ_1 and NZ_2 in this note. The main reason for this interest comes from the fact that the characterization of primitive matrices has nice generalization to primitivity for NZ_2 [45] and Hurwitz primitivity for NZ_1 [42], which we illustrate in subsection 2.1. It worths mentioning that NZ_1 contains the set of stochastic matrices while NZ_2 contains the set of doubly stochastic matrices.

A matrix is an *automaton matrix* if it is a zero-one matrix each row of which contains a unique one. We denote by A the set of all automaton matrices, which is an important subclass of NZ_1. An *automaton* of size n is a subset of $\mathsf{Mat}_n(\mathbb{R}_{\geq 0}) \cap \mathsf{A}$. Also note that a family of $n \times n$ nonnegative integer matrices is nothing but a nondeterministic automaton, namely an arc-labelled digraph. In the literature, an ergodic automaton is also called a *synchronizing automaton*. Černý function, as introduced in subsection 2.2, arises naturally in the study of synchronizing automata and turns out to be crucial in our study of various reachability properties for subsets of NZ_1 and NZ_2.

Let X be a set of nonnegative matrices and let $n \in \mathbb{N}$. We denote by $\mathsf{p}_X(n)$ (resp., $\mathsf{hp}_X(n)$, $\mathsf{e}_X(n)$, $\mathsf{he}_X(n)$) the maximum finite primitive exponent (resp., Hurwitz primitive exponent, ergodic exponent, Hurwitz ergodic exponent) of matrix tuples consisting of some $n \times n$ matrices from X.

2.1. Characterizations via common invariant partitions. A partition π of a nonempty set V is a sequence of nonempty disjoint sets whose union is V; the number of sets in this partition π is called its *size* and is denoted $|\pi|$. We call π *nontrivial* when $|\pi| > 1$. We say that a matrix $A \in \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ *acts on a partition* $\pi = (V_1, \ldots, V_r)$ *of* $[n]$ *subordinate to a permutation* $\sigma \in \mathsf{Sym}_r$ provided $A(V_i, V_j)$ is a zero matrix whenever $j \neq \sigma(i)$. If A acts on π subordinate to the identity permutation, it can be compared with the deck transformation in the theory of covering spaces [26]. For simplicity, we may just say that A *preserves the partition* π when it acts on π subordinate to a permutation. If both A and B preserve the partition π, surely so does their product. Assume that π is a partition of $[n]$ and $\mathcal{A}$ is a set of matrices of order n such that A acts on π subordinate to σ_A for all $A \in \mathcal{A}$. If $\sigma_A \sigma_B = \sigma_B \sigma_A$ for all $A, B \in \mathcal{A}$, then clearly every Hurwitz product of elements from $\mathcal{A}$ will preserve the same partition π. In representation theory, the common invariant subspaces of a family of invertible matrices are the fundamental objects; for general linear operators we can discuss their common invariant cones [41]. For our purpose now, we will see that common invariant partitions will be crucial for understanding the whole picture.

A matrix $A \in \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ is *irreducible* if for every $x, y \in [n]$ there exists a positive integer s such that $A^s(x,y) > 0$. A nonnegative matrix set $\mathcal{A}$ is *irreducible* if the matrix $\sum_{A \in \mathcal{A}} A$ is irreducible. The Perron-Frobenius theorem claims that an irreducible matrix $A \in \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ is primitive if there is no nontrivial partition of $[n]$ on which A acts subordinate to a cyclic permutation. This result has a nice generalization for both primitive matrix sets and Hurwitz primitive matrix sets.

THEOREM 2.1 (Protasov [42, Theorem 1]). *Let* $\mathcal{A} \subseteq \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ *be an irreducible set of matrices belonging to* NZ_1*. Then* $\mathcal{A}$ *is not Hurwitz primitive if and only if we can find a nontrivial partition* π *of* $[n]$ *and* $\sigma_A \in \mathsf{Sym}_{|\pi|}$ *for all* $A \in \mathcal{A}$ *such that* $\sigma_A\sigma_B = \sigma_B\sigma_A$ *and* A *acts on* π *subordinate to* σ_A *for all* $A, B \in \mathcal{A}$*.*

THEOREM 2.2 (Protasov and Voynov [45, Theorem 1]). *Let* $\mathcal{A} \subseteq \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ *be an irreducible set of matrices belonging to* NZ_2*. Then* $\mathcal{A}$ *is primitive if and only if there is no nontrivial partition* π *of* $[n]$ *which is preserved by all elements of* $\mathcal{A}$*.*

The only proof of Theorem 2.1 so far is reported by Protasov [42], which is based on some earlier work of Olesky, Shader and Van den Driessche [37, Theorem 1]. Protasov and Voynov [45] employ geometrical properties of affine operators on polyhedra to give the first proof of Theorem 2.2. There are several later proofs by different authors, using either combinatorial methods [1, 2, 5] or analytic method [57]. We will give a unified proof for both Theorem 2.1 and Theorem 2.2 in section 4. It is a surprise that this unified simple proof is missing in the previous intense study of these characterization results.

To tackle the road coloring problem, Culik, Karhumäki and Kari [14, 29, 30] introduce the concept of stability relation for finite automata. It is named as strong compatibility by Al'pin and Al'pina [1] for general matrix semigroup. Essentially, this is the concept of covering for an arc-labelled digraph [6, 28, 35, 51]. More generally, the concept of equitable partition is of fundamental importance in algebraic combinatorics, which will also play a key role in our work on strongly synchronizing automata [61]. Our unified proof presented in section 4 not only points out that the cornerstones for the theory of Hurwitz primitivity and primitivity, Theorems 2.1 and 2.2, can be easily understood from the point of view of stability relation, but also hints at a possible closer relationship between Hurwitz primitivity and primitivity.

2.2. Exponents and Černý function. According to Gawrychowski and Straszak [20, Theorem 16], there does not exist any constant $\epsilon > 0$ and any polynomial time algorithm that computes $\mathsf{e}(\mathcal{A})$ for all given synchronizing n-state automaton $\mathcal{A}$ within a factor of $n^{1-\epsilon}$, unless P=NP. The *Černý function* c [31, Section 3] [55, Section 3] is nothing but e_{A}, that is,

$$\mathsf{c}(n) = \mathsf{e}_{\mathsf{A}}(n) = \max\{\mathsf{e}(\mathcal{A}) : \ \mathcal{A} \subseteq \mathsf{Mat}_n(\mathbb{R}_{\geq 0}) \text{ is an ergodic automaton}\}$$

for all $n \in \mathbb{N}$. Note that $\mathsf{c}(1) = 1$.

The research on synchronizing automata and the Černý function starts in 1960s [32][33, Chapter IV]. Černý [7, 8] first observes that $(n-1)^2 \leq \mathsf{c}(n) \leq 2^n - n - 1$ for all $n \geq 2$; then he proposes in his talk and in print [9] his famous conjecture.

CONJECTURE 2.3 (Černý). *It holds for all integers* $n \geq 2$ *that* $\mathsf{c}(n) = (n-1)^2$*.*

Two authoritative surveys [31, 55] have expounded in details the work around Conjecture 2.3. We only mention the following upper bounds of the Černý function.

THEOREM 2.4 (Pin [38, Proposition 3.1], Frankl [18, Theorem], Szykuła [52, Theorem 11], Shitov [50, Proposition 7]). *For every integer* $n \geq 2$*,*

$$\mathsf{c}(n) \leq \min\Big\{\frac{n^3-n}{6}, \frac{85059n^3 + 90024n^2 + 196504n - 10648}{511104}, \left(\frac{7}{48} + \frac{2\cdot 15625}{1597536}\right)n^3 + o(n^3)\Big\} = O(n^3).$$

For each $n \in \mathbb{N}$, Blondel, Jungers and Olshevsky [5, Theorem 17,Example 1] obtain the estimate $\frac{n^2}{2} \le \mathsf{p}_{\mathsf{NZ}_2}(n) \le 2\,\mathsf{c}(n) + n - 1 \le O(n^3)$. For every integer $n \ge 2$, Gusev [23, Proposition 5] finds that $\mathsf{hp}(\mathcal{C}_n) \ge \mathsf{he}(\mathcal{C}_n) = (n-1)^2$, where $\mathcal{C}_n$ is the Černý automaton with n states, an automaton consisting of two $n \times n$ matrices. Protasov [44, Conjecture 1] conjectures that $\mathsf{hp}_{\mathsf{NZ}_1}(n)$ is upper bounded by a polynomial of n. We affirm this conjecture of Protasov in subsection 5.1 by showing $\mathsf{he}_{\mathsf{NZ}_1}(n) \le 2\,\mathsf{c}(n) = O(n^3)$ (Theorem 5.3) and $\mathsf{hp}_{\mathsf{NZ}_1}(n) \le 2\,\mathsf{c}(n) + O(n^2) \le O(n^3)$ (Theorem 5.4) for all $n \in \mathbb{N}$.

2.3. Algorithms. For any ergodic m-tuple $\mathcal{A}$ over $\mathsf{NZ}_1(n)$, Protasov designs an algorithm which finds an ergodic word of $\mathcal{A}$ of length $O(n^3)$ within time $O(n^3m)$ [44, Theorem 7,Remark 4]; he also demonstrates an $O(n^3m)$-time algorithm to yield a primitive word of a primitive m-tuple over $\mathsf{NZ}_2(n)$ [44, Theorem 9]. In subsection 5.2, we present an algorithm which finds a Hurwitz ergodic vector of weight $O(n^3)$ for a Hurwitz ergodic m-tuple over $\mathsf{NZ}_1(n)$ in time $O(n^3m^2)$ (Theorem 5.5). This solves a problem posed by Protasov [44, Problem 4]. We also design an algorithm of time complexity $O(n^3m^2)$ which finds a positive Hurwitz primitive vector of weight $O(n^3)$ for any Hurwitz primitive m-tuple over $\mathsf{NZ}_1(n)$ (Theorem 5.6), thus solving another problem raised by Protasov [44, Problem 2].

For any automaton $\mathcal{A} \subseteq \mathsf{Mat}_n(\mathbb{R}_{\ge 0})$ of size m, it is well-known that there exists an algorithm to check whether or not $\mathcal{A}$ is ergodic in time $O(n^2m)$; see [8, Theorem 2], [31, Section 2] and [33, Theorem 15]. For any m-tuple $\mathcal{A}$ over $\mathsf{NZ}_1(n)$, we find that the same idea applies to give an $O(n^2m)$-time algorithm for checking the ergodicity of $\mathcal{A}$ (Theorem 6.1).

For any irreducible m-tuple $\mathcal{A}$ over $\mathsf{NZ}_1(n)$, Protasov finds an $O(n^3m + n^2m^2)$-time algorithm to check if $\mathcal{A}$ is Hurwitz primitive (resp., Hurwitz ergodic) [42, Theorem 2]; if $\mathcal{A}$ is an irreducible automaton, Protasov adapts the above algorithm to check if it is Hurwitz ergodic by spending in total $O(n^2m\log n + n^2m^2)$ arithmetic operations [44, Theorem 12].

Protasov and Voynov [45, Proposition 2] show that Theorem 2.2 leads to an algorithm of deciding primitivity for any given m-tuple $\mathcal{A}$ over $\mathsf{NZ}_2(n)$ in time $O(n^3m)$. Another algorithm of deciding primitivity for such a matrix set is stated (without a proof) by Gusev, Jungers and Pribavkina in [24, Theorem 3.2] whose time-complexity is $O(n^2m)\mathsf{a}(n+m)$, where a is the inverse Ackerman function. We improve these work by presenting a primitivity recognition algorithm for such a matrix set which runs in time $O(n^2m)$ (Theorem 6.2).

2.4. Layout of the paper. The remaining of this note will proceed as follows.

In section 3, adopting the usual approach in combinatorial matrix theory, we explain how to deal with various reachability properties of nonnegative matrix tuples as combinatorial problems about digraphs. Being a warm up in this setting, we derive there $\mathsf{e}(n,1) = \mathsf{he}(n,1) = 1 + (n-2)(n-1)$ (Theorem 3.4) as a graph theory exercise. Note that $\mathsf{e}(n,1) \le 1 + (n-2)(n-1)$ is already reported by Chevalier et al. [11, Corollary 1]; but our new deduction of $\mathsf{e}(n,1) \le 1 + (n-2)(n-1)$ in Theorem 3.4 is more direct and only appeals to a plain fact like [60, Lemma 2.1]. It is interesting that the function $1 + (n-2)(n-1)$ appears in a quantitative version of the road coloring problem [3, Theorems 2 and 7, Conjecture 2].

In section 4, we present a sketch of a proof of Theorems 2.1 and 2.2 from the viewpoint of stable relation.

We devote sections 5 and 6 to recognition algorithms, finding certifying products, and estimating exponents for various reachability properties of matrix sets from NZ_1 and NZ_2. We summarize the newest progress on these issues in Tables 1 and 2. On the one hand, almost all proofs of our new work, possibly excepting that of Lemma 5.2, look to be straightforward modifications of known proofs. On the other hand, we do resolve several open problems and improve existing results. In subsection 5.1, we establish upper bounds for $\mathsf{he}_{\mathsf{NZ}_1}(n)$ and $\mathsf{hp}_{\mathsf{NZ}_1}(n)$, while in subsection 5.2 we present algorithms of finding a Hurwitz primitive (Hurwitz ergodic) vector for a given Hurwitz primitive (Hurwitz ergodic) set of matrices belonging to NZ_1. We finish the paper in section 6 by displaying an algorithm for checking the primitivity property of any given set of matrices belonging to NZ_2.

Table 1
Some results on primitive and Hurwitz primitive m-tuples over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$.

	Primitive		**Hurwitz Primitive**	
Assumption		NZ_2		NZ_1
Time complexity of recognition algorithm	PSPACE-hard [21]	$O(n^2m)$ Theorem 6.2	?	$O(n^3m+n^2m^2)$ [42]
Time complexity of finding a product	PSPACE-hard [21]	$O(n^3m)$ [44]	?	$O(n^3m^2)$ Theorem 5.6
Finding such a shortest product	PSPACE-hard	NP-hard	?	?
Upper bounds of exponents	$\mathsf{p}(n) \leq 3^{\frac{n}{3}(1+\epsilon)}$ when $n \to \infty$ [21]	$\mathsf{p}_{\mathsf{NZ}_2}(n) \leq 2\,\mathsf{c}(n)+n-1$ [5]	$\mathsf{hp}(n,m) \leq m!mn^{m+1}+n^2$ [37]	$\mathsf{hp}_{\mathsf{NZ}_1}(n) \leq 2\,\mathsf{c}(n)+O(n^2)$ Theorem 5.4
Lower bounds of exponents	$\mathsf{p}(n) \geq 3^{\frac{n}{3}(1-\epsilon)}$ when $n \to \infty$ [21]	$\mathsf{p}_{\mathsf{NZ}_2}(n,2) \geq n^2/2$ [5]	$\mathsf{hp}(n,m) \geq n^{m+1}$ [37]	$\mathsf{hp}_{\mathsf{NZ}_1}(n,2) \geq (n-1)^2$ [23]

Table 2
Some results on ergodic and Hurwitz ergodic m-tuples over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$.

	Ergodic		**Hurwitz Ergodic**	
Assumption		NZ_1		NZ_1
Time complexity of recognition algorithm	PSPACE-hard [34]	$O(n^2m)$ Theorem 6.1	?	$O(n^3m+n^2m^2)$ [42]
Time complexity of finding a product	PSPACE-hard [34]	$O(n^3m)$ [44]	?	$O(n^3m^2)$ Theorem 5.5
Finding such a shortest product	PSPACE-hard	NP-hard	?	?
Upper bounds of exponents	$\mathsf{e}(n) \leq 3^{\frac{n}{3}(1+\epsilon)}$ when $n \to \infty$ [46]	$\mathsf{e}_{\mathsf{NZ}_1}(n) \leq \mathsf{c}(n)$	?	$\mathsf{he}_{\mathsf{NZ}_1}(n) \leq 2\,\mathsf{c}(n)$ Theorem 5.3
Lower bounds of exponents	$\mathsf{e}(n) \geq 3^{\frac{n}{3}(1-\epsilon)}$ when $n \to \infty$ [46]	$\mathsf{e}(n,1) = n^2-3n+3$ Theorem 3.4 $\mathsf{e}_{\mathsf{NZ}_1}(n,2) \geq (n-1)^2$ [7]	$\mathsf{he}(n,2) \geq (n-1)^2$ [23]	$\mathsf{he}_{\mathsf{NZ}_1}(n,2) \geq (n-1)^2$ [23]

3. Matrix, digraph, and ergodic exponent. Let $\mathcal{D} = (D_1, \ldots, D_m)$ be an m-tuple of digraphs on the same vertex set V. Let α be a word over $[m]$ of length s. A sequence $(v_0, \ldots, v_s)$ over V is called a *walk of length s from v_0 to v_s labelled by α in $\mathcal{D}$* if (v_{i-1}, v_i) belongs to the arc set of D_{α_i} for all $i \in [s]$. A *nontrivial* walk is a walk of length at least one. A walk $(v_0, \ldots, v_s)$ is *closed* if $v_0 = v_s$. The notation

$x \xrightarrow[\mathcal{D}]{\alpha} y$ means that there exists a walk from x to y in $\mathcal{D}$ labelled by α. Let τ be a vector in $\mathbb{Z}^m_{\geq 0}$. The notation $x \overset{\tau}{\underset{\mathcal{D}}{\dashrightarrow}} y$ means that there exists a word β over $[m]$ such that $x \xrightarrow[\mathcal{D}]{\beta} y$ and $\Psi(\beta) = \tau$. For any two sequences $(x_1, \ldots, x_s)$ and $(y_1, \ldots, y_s)$ over V, we use $(x_1, \ldots, x_s) \xrightarrow[\mathcal{D}]{\alpha} (y_1, \ldots, y_s)$ to denote $x_i \xrightarrow[\mathcal{D}]{\alpha} y_i$ for all $i \in [s]$; we use $(x_1, \ldots, x_s) \overset{\tau}{\underset{\mathcal{D}}{\dashrightarrow}} (y_1, \ldots, y_s)$ to denote $x_i \overset{\tau}{\underset{\mathcal{D}}{\dashrightarrow}} y_i$ for all $i \in [s]$. For any $X \subseteq V$, we say that α *synchronizes* X to a vertex $v \in V$ in $\mathcal{D}$ if $x \xrightarrow[\mathcal{D}]{\alpha} v$ holds for all $x \in X$; we say that τ *Hurwitz synchronizes* X to a vertex $v \in V$ in $\mathcal{D}$ if $x \overset{\tau}{\underset{\mathcal{D}}{\dashrightarrow}} v$ holds for all $x \in X$.

Every matrix $A \in \mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ is associated with a digraph $\mathsf{D}(A)$ on the vertex set $[n]$ in which (x, y) is an arc of $\mathsf{D}(A)$ if and only if $A(x, y) > 0$. For an m-tuple $\mathcal{A} = (A_1, \ldots, A_m)$ over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$, we write $\mathsf{D}(\mathcal{A})$ for the m-tuple of digraphs $\big(\mathsf{D}(A_1), \ldots, \mathsf{D}(A_m)\big)$, which can be viewed as an arc-labelled digraph on $[n]$. Let us recall the following straightforward but useful fact, which says that matrix multiplication is nothing but walks in digraphs.

Lemma 3.1. *Let $\mathcal{A}$ be an m-tuple over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$ and let α be a word over $[m]$. For every $x, y \in [n]$, it holds that $\mathcal{A}_\alpha(x, y) > 0$ if and only if $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha} y$.*

Lemma 3.1 says that various primitivity/ergodicity properties introduced in section 1 are reachability properties for digraphs. Actually, let $\mathcal{A}$ be an m-tuple over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$. Then $\mathcal{A}$ is primitive if there exists a nonempty word α over $[m]$ such that $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha} y$ for all $x, y \in [n]$; $\mathcal{A}$ is Hurwitz primitive if there exists a nonzero vector $\tau \in \mathbb{Z}^m_{\geq 0}$ such that $x \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} y$ for all $x, y \in [n]$; $\mathcal{A}$ is ergodic if there exists a nonempty word α over $[m]$ which synchronizes $[n]$ in $\mathsf{D}(\mathcal{A})$; $\mathcal{A}$ is Hurwitz ergodic if there exists a nonzero vector $\tau \in \mathbb{Z}^m_{\geq 0}$ which Hurwitz synchronizes $[n]$ in $\mathsf{D}(\mathcal{A})$; $\mathcal{A}$ is irreducible if $\mathsf{D}(\mathcal{A})$ is strongly connected, that is, there exists a walk of positive length from x to y for all vertices x and y of $\mathsf{D}(\mathcal{A})$.

Let $\mathcal{D}$ be a tuple of digraphs on a common vertex set V. A *Hamiltonian walk* [4, Section 1.4] in $\mathcal{D}$ is a walk in $\mathcal{D}$ that visits every vertex in V. We write $\mathsf{hamip}_x(\mathcal{D})$ for the length of the shortest Hamiltonian walks in $\mathcal{D}$ starting at $x \in V$ and let $\mathsf{hamip}(\mathcal{D}) = \max_{x \in V} \mathsf{hamip}_x(\mathcal{D})$. We use $\mathsf{hamic}(\mathcal{D})$ to denote the length of the shortest nontrivial closed Hamiltonian walks in $\mathcal{D}$. It surely holds $\mathsf{hamip}(\mathcal{D}) \leq \mathsf{hamic}(\mathcal{D}) - 1$.

Lemma 3.2 (Chang and Tong [10, Theorem 2]). *For every strongly connected digraph D on n vertices, it holds $\mathsf{hamic}(D) \leq \lfloor \frac{(n+1)^2}{4} \rfloor$.*

Lemma 3.3. *Let $\mathcal{A}$ be an irreducible m-tuple over $\mathsf{NZ}_1(n)$. If $\mathcal{A}$ is Hurwitz ergodic, then it is Hurwitz primitive. Moreover, $\mathsf{hp}(\mathcal{A}) \leq \mathsf{he}(\mathcal{A}) + \mathsf{hamip}(\mathsf{D}(\mathcal{A})) \leq \mathsf{he}(\mathcal{A}) + \mathsf{hamic}(\mathsf{D}(\mathcal{A})) - 1 \leq \mathsf{he}(\mathcal{A}) + \lfloor \frac{(n-1)(n+3)}{4} \rfloor$.*

Proof. Take $\tau \in \mathbb{Z}^m_{\geq 0}$ and $x \in [n]$ such that $|\tau| = \mathsf{he}(\mathcal{A})$ and $y \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} x$ for all $y \in [n]$. Since $\mathcal{A}$ is irreducible, we can find a word $\beta = \beta_1 \cdots \beta_s$ over $[m]$ of length $s \leq \mathsf{hamip}(\mathsf{D}(\mathcal{A})) < \infty$ such that there exists an integer $i_z \in [s+1]$ satisfying $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta_1 \cdots \beta_{i_z - 1}} z$ for each $z \in [n]$. Note that we can take $i_x = 1$ and so $\beta_1 \cdots \beta_{i_x - 1}$ is the empty word.

Arbitrarily pick $w, z \in [n]$. As $\mathcal{A} \subseteq \mathsf{NZ}_1(n)$, we can find a vertex $y \in [n]$ such that $w \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta_{i_z}\cdots\beta_s} y$, and thus we have

$$w \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta_{i_z}\cdots\beta_s} y \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} x \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta_1\cdots\beta_{i_z-1}} z.$$

This implies that $\tau + \Psi(\beta)$ is a Hurwitz primitive vector of $\mathcal{A}$. It now follows from Lemma 3.2 that $\mathsf{hp}(\mathcal{A}) \leq |\tau + \Psi(\beta)| = \mathsf{he}(\mathcal{A}) + s \leq \mathsf{he}(\mathcal{A}) + \mathsf{hamip}(\mathsf{D}(\mathcal{A})) \leq \mathsf{he}(\mathcal{A}) + \mathsf{hamic}(\mathsf{D}(\mathcal{A})) - 1 \leq \mathsf{he}(\mathcal{A}) + \lfloor\frac{(n-1)(n+3)}{4}\rfloor$. ∎

THEOREM 3.4. *It holds for each* $n \in \mathbb{N}$ *that* $\mathsf{e}(n,1) = \mathsf{he}(n,1) = 1 + (n-2)(n-1)$.

Proof. The case of $n \leq 2$ is trivial. We thus assume now $n \geq 3$.

Let A be an irreducible n by n ergodic matrix and let $D = \mathsf{D}(A^\top)$. Let C be a shortest closed walk in D of positive length and let c be its length. There exists a vertex x on the cycle C whose out-neighbor in D appear both in C and outside of C. We use X_i to denote the set $\{y : x \overset{i}{\underset{D}{\dashrightarrow}} y\}$. By [60, Lemma 2.1], it holds $2 \leq |X_1| < |X_{1+c}| < |X_{1+2c}| < \cdots < |X_{1+tc}|$, where t is the integer such that $X_{1+(t-1)c} \neq [n]$ and $X_{1+tc} = [n]$. Observe that $c \leq n-1$ and $t \leq n-2$. Henceforth,

$$\mathsf{e}(A) = \mathsf{he}(A) \leq 1 + tc \leq 1 + (n-2)(n-1). \tag{3.1}$$

Let B be an n by n ergodic matrix. Among all strongly connected components of $\mathsf{D}(B)$, there must be exactly one sink component D', namely there is no arc in $\mathsf{D}(B)$ going from D' to the outside of D'. Let k be the number of vertices in D' and let A be the submatrix of B induced by D'. Then $\mathsf{he}(B) = \mathsf{e}(B) \leq n - k + \mathsf{e}(A)$. Considering that $n \geq 3$, we have $n + k \geq 4$ and so $(n+k)(n-k) \geq 4(n-k)$. By (3.1), we now obtain $\mathsf{he}(B) = \mathsf{e}(B) \leq n - k + 1 + (k-2)(k-1) \leq 1 + (n-2)(n-1)$, which implies $\mathsf{e}(n,1) = \mathsf{he}(n,1) \leq 1 + (n-2)(n-1)$.

The n-th Wielandt matrix W_n is the zero-one matrix of order n such that $\mathsf{D}(W_n)$ consists of a closed Hamiltonian walk $1 \to 2 \to \cdots \to n \to 1$ and an extra arc $n \to 2$. By Wielandt's classical observation [59], $\mathsf{hp}(W_n) = (n-1)^2 + 1$. By Lemma 3.3, $\mathsf{e}(W_n) = \mathsf{he}(W_n) \geq \mathsf{hp}(W_n) - \mathsf{hamip}(\mathsf{D}(W_n)) \geq (n-1)^2 + 1 - (n-1) = 1 + (n-2)(n-1)$. This implies that $\mathsf{e}(n,1) = \mathsf{he}(n,1) = \mathsf{e}(W_n) = \mathsf{he}(W_n) = 1 + (n-2)(n-1)$, finishing the proof. ∎

4. Characterizing Hurwitz primitivity and primitivity. Let $\mathcal{A}$ be an m-tuple over $\mathsf{NZ}_1(n)$. Two vertices $x, y \in [n]$ are called *stable* for $\mathcal{A}$, denoted $x \approx_{\mathcal{A}} y$, if for any word α over $[m]$ and for any subset $\{u, v\} \subseteq [n]$ satisfying $(x, y) \xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha} (u, v)$, we can find a word β over $[m]$ which synchronizes $\{u, v\}$ in $\mathsf{D}(\mathcal{A})$. Two vertices $x, y \in [n]$ are called *Hurwitz stable* for $\mathcal{A}$, denoted $x \overset{\mathrm{h}}{\approx}_{\mathcal{A}} y$, if for all vector $\tau \in \mathbb{Z}^m_{\geq 0}$ with $(x, y) \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} (u, v)$, there exists a word τ' over $[m]$ which Hurwitz synchronizes $\{u, v\}$ in $\mathsf{D}(\mathcal{A})$. A key ingredient for our analysis of stability relation is the concept of incompressible set, which is termed an F-clique by Trahtman [54] in the setting of synchronizing automata in honor of Friedman [19]. Two vertices $y, y' \in [n]$ are called *Hurwitz incompressible* for $\mathcal{A}$ provided there is no vector which Hurwitz synchronizes $\{y, y'\}$ in $\mathsf{D}(\mathcal{A})$; similarly, we say that $y, y' \in [n]$ are *incompressible* for $\mathcal{A}$ provided there is no word over $[m]$ which synchronizes $\{y, y'\}$ in $\mathsf{D}(\mathcal{A})$. We call $X \subseteq [n]$ an *incompressible set of* $\mathcal{A}$ or a *Hurwitz incompressible set of* $\mathcal{A}$ if its elements are pairwise incompressible for $\mathcal{A}$ or pairwise Hurwitz incompressible for $\mathcal{A}$, respectively. The stability relation, given by the set of stable pairs, is stable under the action of the semigroup generated by $\mathcal{A}$: If $(x_1, y_1) \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} (x_2, y_2)$ and $x_1 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} y_1$, then $x_2 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} y_2$; if $(x_1, y_1) \xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha} (x_2, y_2)$ and $x_1 \approx_{\mathcal{A}} y_1$, then $x_2 \approx_{\mathcal{A}} y_2$. In some sense, being an incompressible set of $\mathcal{A}$ is also stable under the action of $\mathcal{A}$: If $(x_1, \dots, x_k) \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} (y_1, \dots, y_k)$ and $\{x_1, \dots, x_k\}$ is a Hurwitz incompressible set, then so is $\{y_1, \dots, y_k\}$; if $(x_1, \dots, x_k) \xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha} (y_1, \dots, y_k)$ and $\{x_1, \dots, x_k\}$ is an incompressible set, then so is $\{y_1, \dots, y_k\}$.

Our proof of Lemma 4.1 simply follows the proof of [1, Theorem 2] by Al'pin and Al'pina.

LEMMA 4.1. *Let $\mathcal{A}$ be an m-tuple over $\mathsf{NZ}_1(n)$. If $\mathcal{A}$ is irreducible, then the following hold.*

(1) The Hurwitz stable relation $\overset{h}{\approx}_{\mathcal{A}}$ is an equivalence relation.
(2) The stable relation $\approx_{\mathcal{A}}$ is an equivalence relation.

Proof. (1) The Hurwitz stable relation is clearly a symmetric binary relation.

Assume that $x_1 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} y_1$ and $y_1 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} z_1$ for $x_1, y_1, z_1 \in [n]$. Let $\mathcal{D} = \mathsf{D}(\mathcal{A})$. Let τ be an arbitrary vector in $\mathbb{Z}^m_{\geq 0}$ with $(x_1, y_1, z_1) \overset{\tau}{\underset{\mathcal{D}}{\dashrightarrow}} (x_2, y_2, z_2)$. From $x_1 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} y_1$ we derive the existence of $\phi \in \mathbb{Z}^m_{\geq 0}$ and $u \in [n]$ such that $(x_2, y_2) \overset{\phi}{\underset{\mathcal{D}}{\dashrightarrow}} (u, u)$. Since no matrix from $\mathcal{A}$ has any zero row, there exists $z_3 \in [n]$ such that $z_2 \overset{\phi}{\underset{\mathcal{D}}{\dashrightarrow}} z_3$. In light of $y_1 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} z_1$, there exist $\psi \in \mathbb{Z}^m_{\geq 0}$ and $v \in [n]$ such that $(y_1, z_1) \overset{\tau+\phi}{\underset{\mathcal{D}}{\dashrightarrow}} (u, z_3) \overset{\psi}{\underset{\mathcal{D}}{\dashrightarrow}} (v, v)$. Observe that $x_1 \overset{\tau}{\underset{\mathcal{D}}{\dashrightarrow}} x_2 \overset{\phi}{\underset{\mathcal{D}}{\dashrightarrow}} u \overset{\psi}{\underset{\mathcal{D}}{\dashrightarrow}} v$. We then find that $\phi + \psi$ Hurwitz synchronizes $\{x_2, z_2\}$ in $\mathcal{D}$, and thus $x_1 \overset{\mathrm{h}}{\approx}_{\mathcal{A}} z_1$ follows. This proves that the Hurwitz stable relation is transitive.

Finally, we need to prove the reflexivity of $\overset{\mathrm{h}}{\approx}_{\mathcal{A}}$. Take $y \in [n]$ and $\tau \in \mathbb{Z}^m_{\geq 0}$. Assume that $y \xrightarrow[\mathcal{D}]{\alpha} u_1$ and $y \xrightarrow[\mathcal{D}]{\alpha'} u'_1$ for two words α, α' over $[m]$ with $\Psi(\alpha) = \Psi(\alpha') = \tau$. Let $\{x_1, \dots, x_k\} \subseteq [n]$ be a Hurwitz incompressible set of $\mathcal{A}$ of largest size. As $\mathcal{A}$ is irreducible, we can find a word β such that $x_1 \xrightarrow[\mathcal{D}]{\beta} y$. Let $\phi = \Psi(\beta\alpha) = \tau + \Psi(\beta)$. Since $\mathcal{A}$ falls into NZ_1, we can find $u_2, \dots, u_k$ so that $(x_1, \dots, x_k) \overset{\phi}{\underset{\mathcal{D}}{\dashrightarrow}} (u_1, u_2, \dots, u_k)$ and $(x_1, \dots, x_k) \overset{\phi}{\underset{\mathcal{D}}{\dashrightarrow}} (u'_1, u_2, \dots, u_k)$. Since the $k+1$ elements $u_1, u'_1, u_2, \dots, u_k$ cannot be pairwise Hurwitz incompressible, the only possibility is that u_1 and u'_1 can be Hurwitz synchronized. This proves $y \overset{\mathrm{h}}{\approx}_{\mathcal{A}} y$, as wanted.

(2) The proof is similar to the proof of (1). ☐

We recall a basic observation in the study of synchronizing phenomena, which indeed goes back to the very beginning of this subject; see [8, Theorem 2] and [33, Theorem 15].

LEMMA 4.2. *Let $\mathcal{A}$ be an m-tuple over $\mathsf{NZ}_1(n)$ and let $\mathcal{D} = \mathsf{D}(\mathcal{A})$.*

(1) Assume that for every $x, y \in [n]$, there exists a vector $\tau \in \mathbb{Z}^m_{\geq 0}$ such that τ

Hurwitz synchronizes $\{x, y\}$ in $\mathcal{D}$. Then $\mathcal{A}$ has a Hurwitz ergodic vector.

(2) Assume that for every $x, y \in [n]$, there exists a word α over $[m]$ such that α synchronizes $\{x, y\}$ in $\mathcal{D}$. Then $\mathcal{A}$ possesses an ergodic word.

Proof. (1) Every singleton set inside $[n]$ can be trivially Hurwitz synchronized. So, to finish the proof, we take a proper subset X of $[n]$ and an element $z \in [n] \setminus X$, and aim to show that $X \cup \{z\}$ can be synchronized in $\mathcal{A}$ under the assumption that $\phi \in \mathbb{Z}^m_{\geq 0}$ synchronizes X to $y \in [n]$.

Since $\mathcal{A} \subseteq \mathsf{NZ}_1$, there exists $z' \in [n]$ such that $z \overset{\phi}{\underset{\mathcal{D}}{\dashrightarrow}} z'$. By our assumption, there exists a vector ψ which Hurwitz synchronizes $\{z', y\}$. Then $\phi + \psi$ Hurwitz synchronizes $X \cup \{z\}$ in $\mathcal{D}$, as desired.

(2) The proof is analogous to the proof of (1). ∎

Lemma 4.3. *Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an irreducible m-tuple over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$.*

(1) Assume that $A_1, \ldots, A_m \in \mathsf{NZ}_1$. Then, $\mathcal{A}$ is Hurwitz primitive if and only if $u \overset{h}{\approx}_{\mathcal{A}} v$ for all $u, v \in [n]$.

(2) Assume that $A_1, \ldots, A_m \in \mathsf{NZ}_2$. Then, $\mathcal{A}$ is primitive if and only if $u \approx_{\mathcal{A}} v$ for all $u, v \in [n]$.

Proof. For both (1) and (2), it is enough to prove the backward direction.

(1) Assuming that $u \overset{\mathrm{h}}{\approx}_{\mathcal{A}} v$ for all $u, v \in [n]$, Lemma 4.2 then claims that $\mathcal{A}$ is Hurwitz ergodic. By Lemma 3.3, $\mathcal{A}$ is Hurwitz primitive.

(2) Let $\mathcal{B}$ be the m-tuple $(A_1^\top, \ldots, A_m^\top)$. By Lemma 4.1, the stable relation $\approx_{\mathcal{B}}$ gives a partition π of $[n]$. Since $\mathcal{A} \in \mathsf{NZ}_2$, we see that both $\mathcal{A}$ and $\mathcal{B}$ preserve the partition π. Since we have assumed that the stable relation $\approx_{\mathcal{A}}$ is $[n] \times [n]$, we see that $|\pi| = 1$ and so $\approx_{\mathcal{A}} = \approx_{\mathcal{B}} = [n] \times [n]$. By Lemma 4.2, there exists a word α which synchronizes $[n]$ to a vertex $x \in [n]$ in $\mathsf{D}(\mathcal{A})$ and there exists a word β which synchronizes $[n]$ to a vertex $y \in [n]$ in $\mathsf{D}(\mathcal{B})$. Since $\mathcal{A}$ is irreducible, there exists a word γ over $[m]$ such that $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\gamma} y$. Let β' be the reversal of β. It is easy to see that

$$w \xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha} x \xrightarrow[\mathsf{D}(\mathcal{A})]{\gamma} y \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta'} z$$

for all $w, z \in [n]$. That is, $\mathcal{A}$ is primitive. ∎

Proof of Theorem 2.1. Immediate from Lemma 4.1 (1) and Lemma 4.3 (1). ∎

Proof of Theorem 2.2. By Lemma 4.1 (2) and Lemma 4.3 (2). ∎

5. Hurwitz ergodicity and Hurwitz primitivity.

5.1. Exponents. We start with a folklore relation between ergodic exponent and the Černý function [5, 58].

Lemma 5.1. *Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an m-tuple over $\mathsf{NZ}_1(n)$. If $\mathcal{A}$ is ergodic, then $\mathsf{e}(\mathcal{A}) \leq \mathsf{c}(n)$.*

Proof. Let $\mathcal{B}$ be the set

$$\bigcup_{i \in [m]} \{B \in \mathsf{A} : \quad B(x, y) > 0 \text{ implies } A_i(x, y) > 0 \text{ for all } x, y \in [n] \ \}.$$

Notice that $\mathcal{B}$ is simply the set of n by n automaton matrices whose support is contained in the support of any one of $\mathcal{A}$. It surely holds that $\mathcal{B}$ is ergodic and $\mathsf{e}(\mathcal{A}) \leq \mathsf{e}(\mathcal{B}) \leq \mathsf{c}(n)$. ∎

FIG. 1. *The arc-labelled digraphs corresponding to* $\mathcal{A}$ *and* $\mathcal{A}^{(2)}$*, where* $\mathcal{A} = (A_1, A_2)$ *and* $\mathcal{A}^{(2)} = (A_1, A_2, A_3 = A_1A_2 + A_2A_1)$*.*

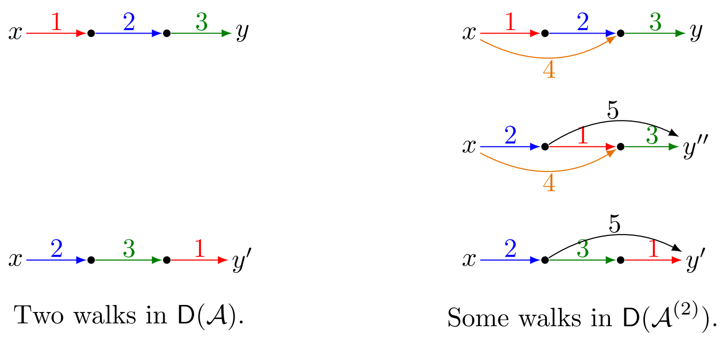

FIG. 2. *Let* $\mathcal{A} = (A_1, A_2, A_3)$ *and* $\mathcal{A}^{(2)} = (A_1, A_2, A_3, A_4 = A_1A_2 + A_2A_1, A_5 = A_1A_3 + A_3A_1, A_6 = A_2A_3 + A_3A_2)$*. Observe that* $x \xrightarrow[\mathsf{D}(\mathcal{A}^{(2)})]{(4,3)} y$, $x \xrightarrow[\mathsf{D}(\mathcal{A}^{(2)})]{(4,3)} y''$, $x \xrightarrow[\mathsf{D}(\mathcal{A}^{(2)})]{(2,5)} y''$ *and* $x \xrightarrow[\mathsf{D}(\mathcal{A}^{(2)})]{(4,3)} y'$*. These imply that* $y \approx_{\mathcal{A}^{(2)}} y''$ *and* $y'' \approx_{\mathcal{A}^{(2)}} y'$*, yielding* $y \approx_{\mathcal{A}^{(2)}} y'$*.*

For any two words $\beta = \beta_1 \cdots \beta_\ell$ and $\beta' = \beta'_1 \cdots \beta'_\ell$, we say that β and β' *differ by a swapping at* $i \in [\ell - 1]$ if $\beta_i = \beta'_{i+1}$, $\beta_{i+1} = \beta'_i$ and $\beta_j = \beta'_j$ for all $j \in [\ell] \setminus \{i, i+1\}$. Since the symmetric group on $[\ell]$ is generated by transpositions of successive numbers, we know that for any two words β and β' of the same Parikh vector, we can find a sequence of words $\beta(1) = \beta, \beta(2), \ldots, \beta(t-1), \beta(t) = \beta'$ such that $\beta(k)$ and $\beta(k+1)$ differ by a swapping for all $k \in [t-1]$. Let $\mathcal{A} = (A_1, \ldots, A_m)$ be an m-tuple over $\mathsf{Mat}_n(\mathbb{R}_{\geq 0})$. We reserve the notation $\mathcal{A}^{(2)}$ for the set

$$\{A_i, A_iA_j + A_jA_i : \ i, j \in [m]\};$$

see Figure 1 for an illustration. We are now ready to establish Lemma 5.2, which presents a reduction from Hurwitz ergodic sets of matrices to simply ergodic sets of matrices. Note that our work in section 4 displays the similarity in primitivity and Hurwitz primitivity, while Lemma 5.2 exposes a strong link between ergodicity and Hurwitz ergodicity.

LEMMA 5.2. *Let* $\mathcal{A}$ *be a Hurwitz ergodic* m*-tuple over* $\mathsf{NZ}_1(n)$*. Then* $\mathcal{A}^{(2)}$ *is ergodic and* $\mathsf{he}(\mathcal{A}) \leq 2\,\mathsf{e}(\mathcal{A}^{(2)}) \leq 2\,\mathsf{c}(n)$*.*

Proof. Since every matrix in $\mathcal{A}^{(2)}$ is a Hurwitz product over $\mathcal{A}$ of length at most 2, it holds that $\mathsf{he}(\mathcal{A}) \leq 2\,\mathsf{e}(\mathcal{A}^{(2)})$. Under the assumption that $\mathcal{A}^{(2)}$ is ergodic, Lemma 5.1 gives $\mathsf{e}(\mathcal{A}^{(2)}) \leq \mathsf{c}(n)$. Therefore, our task is to show that $\mathcal{A}^{(2)}$ is ergodic.

We first consider the case that $\mathcal{A}$ is irreducible. Fix $x \in [n]$ and take arbitrarily $(y, y') \in [n] \times [n]$. We get from Lemma 3.3 that $\mathcal{A}$ is Hurwitz primitive and so there exists $\tau \in \mathbb{Z}^m_{\geq 0}$ such that $(x, x) \overset{\tau}{\underset{\mathsf{D}(\mathcal{A})}{\dashrightarrow}} (y, y')$. Assume that $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta} y$ and $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta'} y'$ for two words β and β' having the same Parikh vector τ. We then pick a sequence of words $\beta(1)=\beta,\beta(2),\dots,\beta(t-1),\beta(t)=\beta'$ such that $\beta(k)$ and $\beta(k+1)$ differ by a swapping for all $k\in[t-1]$. Since $\mathcal{A}\subseteq \mathsf{NZ}_1$, we can assume $x \xrightarrow[\mathsf{D}(\mathcal{A})]{\beta(k)} y(k)$ for all $k\in[t]$, where $y(1)=y$ and $y(k)=y'$. Accordingly, one can find a word $\gamma(k)$ such that $(x,x) \xrightarrow[\mathsf{D}(\mathcal{A}^{(2)})]{\gamma(k)} (y(k),y(k+1))$ for each $k\in[t-1]$. Since $\mathcal{A}^{(2)}\supseteq\mathcal{A}$ and $\mathcal{A}$ is irreducible, we know that $\mathcal{A}^{(2)}$ is irreducible. By Lemma 4.1 (2), we thus conclude that $x\approx_{\mathcal{A}^{(2)}} x$ and $y=y(1)\approx_{\mathcal{A}^{(2)}}\cdots\approx_{\mathcal{A}^{(2)}} y(k)=y'$. We refer the reader to Figure 2 for the simple idea behind this line of argument. An application of Lemma 4.2 (2) now yields that $\mathcal{A}^{(2)}$ is ergodic.

We next turn to the case that $\mathcal{A}$ is not irreducible. For any subset X of $[n]$, we write $\mathcal{A}[X]$ for the m-tuple $(A_1[X],\dots,A_m[X])$, where, for each $i\in[m]$, $A_i[X]$ is the submatrix of A_i induced by X. Since $\mathcal{A}$ is Huiwitz ergodic, we can find a strongly connected component X of $\mathsf{D}(\mathcal{A})$ such that from every $y\in[n]$ there exists a walk of $\mathsf{D}(\mathcal{A})$ leading into X. Observe that $\mathcal{A}[X]\subseteq\mathsf{NZ}_1$. Let k be the size of X and enumerate $[n]\setminus X$ as $y_1,\dots,y_{n-k}$. For every $i\in[n-k]$, there exists a walk $\alpha(i)$ from y_i to some vertex in X. Then $(y_1,\dots,y_n)\xrightarrow[\mathsf{D}(\mathcal{A})]{\alpha}(x_1,\dots,x_n)$, where $x_i\in X$ for all $i\in[n]$ and $\alpha=\alpha(1)\alpha(2)\cdots\alpha(n-k)$. On the other hand, since $\mathcal{A}[X]$ is irreducible and Hurwitz ergodic, we already know above that $\mathcal{A}[X]^{(2)}$ possesses an ergodic word α'. It follows that $\mathcal{A}^{(2)}$ has $\alpha\alpha'$ as an ergodic word, as was to be shown. ∎

THEOREM 5.3. *For all* $n\in\mathbb{N}$, $\mathsf{he}_{\mathsf{NZ}_1}(n)\le 2\,\mathsf{c}(n)=O(n^3)$.

Proof. Apply Theorem 2.4 and Lemma 5.2. ∎

THEOREM 5.4. *For all* $n\in\mathbb{N}$, $\mathsf{hp}_{\mathsf{NZ}_1}(n)\le 2\,\mathsf{c}(n)+\lfloor\frac{(n-1)(n+3)}{4}\rfloor=O(n^3)$.

Proof. This follows directly from Lemma 3.3 and Theorem 5.3. ∎

5.2. Finding Hurwitz ergodic vector and Hurwitz primitive vector. Our proofs of Theorems 5.3 and 5.4 are constructive and the idea there will enable us to find a short Hurwitz primitive (Hurwitz ergodic) vector in polynomial time, thus providing an answer to [44, Problems 2 and 4].

Algorithm 5.1 Find a Hurwitz ergodic vector for a set of matrices belonging to NZ_1.

Require: Input a Hurwitz ergodic m-tuple $\mathcal{A}$ over $\mathsf{NZ}_1(n)$.

1: Construct an m-tuple $\mathcal{B}=(B_1,\dots,B_m)$ over $\mathsf{Mat}_n(\mathbb{R}_{\ge 0})$ where $B_i(x,y)=\begin{cases}1, & \text{if } A_i(x,y)>0,\\ 0, & \text{otherwise,}\end{cases}$ for all $i\in[m]$ and $x,y\in[n]$.
2: Construct the matrix set $\mathcal{C}=\mathcal{B}^{(2)}$ and let $\ell=|\,\mathcal{C}\,|$.
3: Find a map f from $[\ell]$ to $\binom{[m]}{1}\cup\binom{[m]}{2}$ such that for every $k\in[\ell]$, either $C_k=B_i=B_j$ or $C_k=B_iB_j+B_jB_i$, where $f(k)=\{i,j\}$.
4: Find an ergodic word α of $\mathcal{C}$ of length $s=O(n^3)$.
5: Calculate $\tau\in\mathbb{Z}^m_{\ge 0}$ where $\tau(i)=|\{j\in[s]:\ i\in f(\alpha_j)\}|$ for each $i\in[m]$.
6: **return** τ.

THEOREM 5.5. *For any Hurwitz ergodic m-tuple $\mathcal{A}$ over* $\mathsf{NZ}_1(n)$, *Algorithm 5.1 finds a Hurwitz ergodic vector τ for $\mathcal{A}$ with $|\tau|=O(n^3)$ in time $O(n^3m^2)$.*

Proof. The time complexity of obtaining $\mathcal{B}$ is $O(n^2m)$. In order to get $\mathcal{C}$ and f, it suffices to do $O(m^2)$ multiplications of two matrices of order n, and this work costs time $O(n^3m^2)$. By Lemma 5.2, $\mathcal{C}$ is ergodic. There is an algorithm to find an ergodic product α of length $O(n^3)$ over $\mathcal{C}$ in time $O(n^3m^2)$; see for example [44, Algorithm 2, Theorem 9]. Since the length of α is $O(n^3)$, one can calculate the vector τ in time $O(n^3m)$. Recall that every matrix in $\mathcal{C}$ either belongs to $\mathcal{B}$ or equals to $B_iB_j + B_jB_i$ for some $i, j \in [m]$. Therefore, it holds

$$\mathcal{B}^\tau = \sum_{\Psi(\beta)=\tau} \mathcal{B}_\beta \geq \mathcal{C}_\alpha > 0,$$

which then ensures that $\mathcal{A}^\tau > 0$. Also note that $|\tau| \leq 2s = O(n^3)$. Finally, we can check that the running time of Algorithm 5.1 is $O(n^3m^2)$. ∎

THEOREM 5.6. *There exists an algorithm to find a Hurwitz primitive vector $\tau \in \mathbb{Z}^m_{\geq 0}$ with $|\tau| = O(n^3)$ in time $O(n^3m^2)$ for any given Hurwitz primitive m-tuple $\mathcal{A}$ over $\mathsf{NZ}_1(n)$.*

Proof. By Theorem 5.5, in time $O(n^3m^2)$ one obtains a vector $\phi \in \mathbb{Z}^m_{\geq 0}$ such that $\mathcal{A}^\phi$ has a positive column, say the x-th column, and $|\phi| = O(n^3)$. Let $\mathcal{D} = \mathsf{D}(\mathcal{A})$ be the arc-labelled digraph on $[n]$. Because $\mathcal{A}$ is Hurwitz primitive, $\mathcal{D}$ has to be strongly connected. Within time $O(n^2m)$ we can find a Hamiltonian walk H of $\mathcal{D}$ starting at x and of length $O(n^2)$: List all vertices of $\mathcal{D}$ as $x_1, \ldots, x_n$ where $x_1 = x$; find a shortest path from x_i to x_{i+1} for $i \in [n-1]$; concatenate all these paths. Let $\psi \in \mathbb{Z}^m_{\geq 0}$ be the vector such that $\psi(k)$ equals the number of arcs with label k, counted with multiplicity, appearing in the Hamiltonian walk H for all $k \in [m]$. Let $\tau = \phi + \psi$. Following the proof of Lemma 3.3, we see that τ is a Hurwitz primitive vector of $\mathcal{A}$. Meanwhile, $|\tau| = O(n^3)$ is trivial to see. ∎

6. Ergodicity and primitivity. The digraph H used in the proof of the subsequent Theorem 6.1 appears already in the proof of Voynov [58, Theorem 1] for $\mathsf{p}_{\mathsf{NZ}_2}(n) \leq \frac{n^3+n^2}{2} - 2n + 1$. Al'pin and Al'pina [2, Section 4] construct an analogous digraph in their algorithm for finding the maximum partition preserved by any given irreducible set of matrices belonging to NZ_2. It is a pleasure that Theorem 6.2, our improvement of corresponding results from [24, 45], just rests on these old simple ideas.

THEOREM 6.1. *For any m-tuple $\mathcal{A}$ over $\mathsf{NZ}_1(n)$, there exists an algorithm of time complexity $O(n^2m)$ which checks whether or not $\mathcal{A}$ is ergodic.*

Proof. Construct a digraph H on the vertex set $[n] \times [n]$ such that there is an arc from (x, y) to (x', y') in H if and only if there exists $A \in \mathcal{A}$ satisfying $A(x, x')A(y, y') > 0$. Let $V_1 = \{(z, z) : z \in [n]\}$ be the diagonal of $[n] \times [n]$ and $V_2 = ([n] \times [n]) \setminus V_1$.

We claim that $\mathcal{A}$ is ergodic if and only if for all vertices $(x, y) \in V_2$ there exists a walk in H going from (x, y) into V_1. Indeed, the 'only if' part is simply due to Lemma 3.1, while the 'if' part is guaranteed by Lemma 3.1 and Lemma 4.2 (2).

Using breadth-first search [13, Section 22.2], it costs time $O(n^2m)$ to check whether or not all vertices from V_2 can reach V_1 in H. ∎

THEOREM 6.2. *For any m-tuple $\mathcal{A}$ over $\mathsf{NZ}_2(n)$, there exists an $O(n^2m)$-time algorithm to determine whether or not $\mathcal{A}$ is primitive.*

Proof. By virtue of Lemma 4.2 (2) and Lemma 4.3 (2), saying that $\mathcal{A}$ is primitive amounts to saying that it is both irreducible and ergodic. Using the classical algorithm of Tarján [53, Theorem 13], we can check whether or not $\mathcal{A}$ is irreducible in time $O(n^2m)$. By Theorem 6.1, we can determine whether or not $\mathcal{A}$ is ergodic in time $O(n^2m)$. ∎

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