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Homogeneity of transformation semigroups*

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Abstract

Let S be a transformation semigroup acting on a set Ω. The action of S on Ω can be naturally extended to be an action on all subsets of Ω. We say that S is ℓ-homogeneous provided it can send A to B for any two (not necessarily distinct) ℓ-subsets A and B of Ω. On the condition that $k \leq \ell < k + \ell \leq |\Omega|$, we show that every ℓ-homogeneous transformation semigroup acting on Ω must be k-homogeneous. We report other variants of this result and suggest a matroid framework for further research along the same direction.

Keywords: automaton, Grassmannian, inclusion operator, permutation group, phase space, valued poset.

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1 Phase spaces of transformation semigroups

Let Γ be a **digraph**, namely a pair consisting of its vertex set $\mathrm{V}(\Gamma)$ and arc set $\mathrm{E}(\Gamma) \subseteq \mathrm{V}(\Gamma) \times \mathrm{V}(\Gamma)$. We call Γ **symmetric** if $(u, v) \in \mathrm{E}(\Gamma)$ if and only if $(v, u) \in \mathrm{E}(\Gamma)$. For any $A \subseteq \mathrm{V}(\Gamma)$, we adopt the notation $\Gamma[A]$ for the subdigraph of Γ induced by A which has vertex set A and arc set $\mathrm{E}(\Gamma) \cap (A \times A)$. The number of weakly connected components and the number of strongly connected components of Γ will be dubbed $\mathrm{wcc}(\Gamma)$ and $\mathrm{scc}(\Gamma)$, respectively.

For a set Ω, all maps from Ω to itself form the set Ω^Ω. For each $g \in \Omega^\Omega$ and $\alpha \in \Omega$, we write αg for the image of α under the map g. The composition of maps provides an associative product on the set Ω^Ω and thus turns it into a monoid, namely a semigroup with a multiplicative unit. We call this monoid the **full transformation monoid** on Ω and denote it by $\mathrm{T}(\Omega)$. A subset of $\mathrm{T}(\Omega)$ which is closed under map composition, whether or not it contains the identity map on Ω, is called a **transformation semigroup** acting on Ω. Let S be a transformation semigroup on Ω. We say that S is **transitive on a set** $A \subseteq \Omega$ if for

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every $\alpha, \beta \in A$ we can find $g \in S$ such that $\alpha g = \beta$; we call S **transitive** if S is transitive on Ω. If the transformation semigroup S is generated by a set $G \subseteq \Omega^\Omega$, namely S consists of products of elements of G of positive length, we call (S, G) a **deterministic automaton** on Ω [41, §1]. The **phase space** of an automaton (S, G) on Ω, denoted by $\Gamma(S, G)$, is the digraph with vertex set Ω and arc set $\{(\alpha, \alpha g) : \alpha \in \Omega, g \in G\}$. When Ω has at least two elements, the claim that S is transitive is equivalent to the claim that $\Gamma(S, G)$ is strongly connected for any generator set G of S. We write $\Gamma(S, S)$ simply as $\Gamma(S)$ and note that each strongly/weakly connected component of $\Gamma(S)$ coincides with a strongly/weakly connected component of $\Gamma(S, G)$ for any generator set G of S. For all work in this paper, we can simply focus on $\Gamma(S)$ instead of considering $\Gamma(S, G)$ for any specific generator set G. We emphasize $\Gamma(S, G)$ from the phase space viewpoint here to highlight the connection between semigroup theory and automaton theory, and to indicate the role played by the choice of G in some problems related to various distance functions on the phase space, including Černý conjecture. For any set Ω, a subset of $\mathrm{T}(\Omega)$ forms a **permutation group** on Ω whenever it is a transformation semigroup and each element has an inverse in it, namely it is a set of bijective transformations of Ω and is closed under compositions and taking inverses. Permutation groups correspond to reversible deterministic automata.

Let Ω be a set. We follow the common practice to use 2^Ω for the power set of Ω. For each $g \in \mathrm{T}(\Omega)$, let $\overline{g}$ be the element in $\mathrm{T}(2^\Omega)$ that sends each $A \in 2^\Omega$ to $A\overline{g} \doteq \{ag : a \in A\}$. More generally, for each $G \subseteq \mathrm{T}(\Omega)$, $\overline{G}$ refers to the set $\{\overline{g} : g \in G\}$. For any transformation semigroup S on Ω and any generator set G of S, $\overline{S}$ is known to be the **powerset transformation semigroup of** S acting on 2^Ω and $(\overline{S}, \overline{G})$ is known to be the **powerset automaton** of (S, G). It may be interesting to iterate the powerset automaton construction and examine the evolution of the phase spaces of the resulting automata. Homogeneity is a natural concept from this point of view. For any positive integer $k \leq |\Omega|$, a transformation semigroup S on Ω is k**-homogeneous** if the transformation semigroup $\overline{S}$ is transitive on $\binom{\Omega}{k}$.

Let A and Ω be two sets with $A \subseteq \Omega$. For any $g \in \Omega^\Omega$, write $g|_A$ for the restriction of g on A; for any $f \in A^A$, write $f|^\Omega$ for the lift of f to Ω, which is defined to be the element $g \in \Omega^\Omega$ such that $g|_A = f$ and g fixes every element of $\Omega \setminus A$. Let S be a transformation semigroup on Ω. The **stabiliser permutation group** of (S, A) is the permutation group $S_A \doteq \{g|_A : g \in S, A\overline{g} = A\}$ acting on A. The **relative transformation semigroup** of (S, A) is the transformation semigroup $\tilde{S}_A \doteq \{g|_A : g \in S, A\overline{g} \subseteq A\}$ acting on A. Note that $\tilde{S}_A$ may not be transitive on A even when S is transitive on A.

Theorem 1.1. *Let Ω be a set and let k and ℓ be two positive integers such that $k \leq \ell < k + \ell \leq |\Omega|$. Let S be a transformation semigroup on Ω and let Γ be the phase space of $\overline{S}$.*

(1) *If S is ℓ-homogeneous, then it is k-homogeneous.*

(2) *If Ω is a finite set, then* $\mathrm{wcc}(\Gamma[\binom{\Omega}{k}]) \leq \mathrm{wcc}(\Gamma[\binom{\Omega}{\ell}])$.

(3) *Let $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell+1}$. If Ω is finite and S is $(\ell+1)$-homogeneous, then* $\mathrm{scc}(\Gamma(S_A)) = \mathrm{wcc}(\Gamma(S_A)) \leq \mathrm{wcc}(\Gamma(S_B)) = \mathrm{scc}(\Gamma(S_B))$.

Conjecture 1.2. *Take a finite set Ω and two positive integers k and ℓ such that $k \leq \ell < k + \ell \leq |\Omega|$. Let S be an ℓ-homogeneous transformation semigroup acting on Ω. For any $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$, it holds* $\mathrm{wcc}(\Gamma(\tilde{S}_A)) \leq \mathrm{wcc}(\Gamma(\tilde{S}_B))$.

When restricting to permutation groups, the results in Theorem 1.1 are all known more than 40 years ago: (1) for an infinite set Ω was discovered by Brown [7, Corollary 1]; (1) and (2) for a finite set Ω were derived by Livingstone and Wagner [23, Theorem 1]; (3), as well as Conjecture 1.2 for permutation groups, was proved by Cameron [10, Proposition 2.3]. Let G be a group acting on a finite set Ω and let k and ℓ be two positive integers such that $k \leq \ell < k + \ell \leq |\Omega|$. By Theorem 1.1 (2), or more precisely Livingstone-Wagner Theorem [23, Theorem 1], we know that the number of $\overline{G}$-orbits on $\binom{\Omega}{\ell}$ is no less than the number of $\overline{G}$-orbits on $\binom{\Omega}{k}$. As an improvement of this fact, Siemons [34, Corollary 4.3] found a natural linear space whose dimension equals the integer $\mathrm{wcc}(\Gamma[\binom{\Omega}{\ell}]) - \mathrm{wcc}(\Gamma[\binom{\Omega}{k}])$ and he [34, Theorem 4.2] even obtained an algorithm to reconstruct the $\overline{G}$-orbits on $\binom{\Omega}{k}$ from the information on the $\overline{G}$-orbits on $\binom{\Omega}{\ell}$ without reference to the group G.

Example 1.3. Let Ω be a set with a linear order $\prec$ on it. A map $g \in \Omega^\Omega$ is order-preserving with respect to $\prec$ provided αg is not bigger than βg in $\prec$ whenever α is not bigger than β in $\prec$. Let S be the monoid consisting of all order-preserving maps on Ω with respect to the given linear order $\prec$. It is easy to see that S is ℓ-homogeneous for all $\ell \leq |\Omega|$. Note that the only permutation contained in S is the identity map in case that Ω is a finite set. This suggests that you may not be able to read Theorem 1.1 directly from those known facts on permutation groups.

Question 1.4. Let Ω be a finite set and let k and ℓ be two positive integers such that $k \leq \ell < k + \ell \leq |\Omega|$. Let S be a transformation semigroup on Ω and let Γ be the phase space of $\overline{S}$.

(1) Is there a counterpart of [34, Corollary 4.3] which explains the nonnegativeness of the integers $\mathrm{wcc}(\Gamma[\binom{\Omega}{\ell}]) - \mathrm{wcc}(\Gamma[\binom{\Omega}{k}])$ and $\mathrm{scc}(\Gamma(S_B)) - \mathrm{scc}(\Gamma(S_A))$ for any $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$?

(2) Is there an algorithm to determine the weakly connected components of $\Gamma[\binom{\Omega}{k}]$ from the weakly connected components of $\Gamma[\binom{\Omega}{\ell}]$ without reference to the transformation semigroup S?

(3) Is it true that $\mathrm{scc}(\Gamma[\binom{\Omega}{k}]) \leq \mathrm{scc}(\Gamma[\binom{\Omega}{\ell}])$?

Neumann [27] asked whether every λ-homogeneous permutation group is θ-homogeneous for all cardinals $\lambda > \theta \geq \aleph_0$. Assuming Martin's Axiom, Shelah and Thomas [33] gave a negative answer to it. Hajnal [18] supplied an example to show that 2^θ-homogeneity does not imply θ-homogeneity. For each statement in Theorem 1.1, Conjecture 1.2, and Question 1.4, it is interesting to see whether or not they hold when the set Ω is infinite. We are also wondering if the rich theory on oligomorphic permutation groups [11] should have a counterpart for transformation semigroups.

For any two positive integers k and ℓ, we say that a transformation semigroup S acting on Ω is (k, ℓ)-**homogeneous** provided for every $A \in \binom{\Omega}{k}$ and $B \in \binom{\Omega}{\ell}$ we can find $g \in S$ such that $A\overline{g} \cap B \in \{A\overline{g}, B\}$. Araújo and Cameron [2] have studied (k, ℓ)-homogeneous permutation groups. Beyond the homogeneity property discussed so far, there has been an active study of those permutation groups which are transitive on the set of all ordered or unordered partitions of a set of a given shape [1, 15, 24, 26]. There are many ways to define corresponding properties for transformation semigroups. A systematic study of the extension of the results on permutation groups to transformation semigroups should be fruitful.

For a linear space V, the Grassmannian $\mathrm{Gr}(V)$ is $\cup_{k=0}^{\infty} \mathrm{Gr}(k, V)$, where $\mathrm{Gr}(k, V)$ is the set of all k-dimensional linear subspaces of V for each nonnegative integer k. Let Ω be a finite set and F be a field. We mention that $\mathrm{Gr}(k, F^{\Omega})$ is a q-analogue of $\binom{\Omega}{k}$ and their relationship is like the one between Johnson graphs and Grassmann graphs [28]. For each prime power q and positive integer n, we write $\mathbb{F}_q$ for the q-element finite field and write $\mathrm{Mat}_n(\mathbb{F}_q)$ for the multiplicative semigroup of all n by n matrices over $\mathbb{F}_q$. Let k, ℓ and n be three nonnegative integers such that $k \leq \ell \leq n$. The set of all linear subspaces of $\mathbb{F}_q^n$ is denoted by $\mathrm{Gr}(\mathbb{F}_q^n) = \mathcal{P}_{q,n}$ and the set of all dimension-k linear subspaces of $\mathbb{F}_q^n$ is denoted by $\mathrm{Gr}(k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^k$. The set of all affine subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{A}_{q,n}$ and the set of all dimension-k affine subspaces of $\mathbb{F}_q^n$ is denoted by $\mathcal{A}_{q,n}^k$. We are ready to display Theorem 1.5, a q-analogue of Theorem 1.1. If the semigroup $S \leq \mathrm{Mat}_n(\mathbb{F}_q)$ is a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$, Theorem 1.5 was already reported by Stanley in 1982 [36, Corollary 9.9].

Theorem 1.5. *Let k, ℓ and n be three positive integers such that $k < \ell \leq n-k$ and let $\mathbb{F}_q$ be the finite field of q elements. Let $S \leq \mathrm{Mat}_n(\mathbb{F}_q)$ be a linear transformation semigroup acting on $\mathbb{F}_q^n$. For each $g \in S$, write $g^{\mathcal{P}}$ for $\overline{g}|_{\mathcal{P}_{q,n}}$ and write $g^{\mathcal{A}}$ for $\overline{g}|_{\mathcal{A}_{q,n}}$. Let $S^{\mathcal{P}}$ be the transformation semigroup $\{g^{\mathcal{P}} : g \in S\}$ acting on $\mathcal{P}_{q,n}$ and let $S^{\mathcal{A}}$ be the transformation semigroup $\{g^{\mathcal{A}} : g \in S\}$ acting on $\mathcal{A}_{q,n}$. We use $\Gamma_{\mathcal{P}}$ and $\Gamma_{\mathcal{A}}$ for the phase spaces of $S^{\mathcal{P}}$ and $S^{\mathcal{A}}$, respectively.*

(1) $\mathrm{wcc}(\Gamma_{\mathcal{P}}[\mathcal{P}_{q,n}^k]) \leq \mathrm{wcc}(\Gamma_{\mathcal{P}}[\mathcal{P}_{q,n}^{\ell}])$.

(2) $\mathrm{wcc}(\Gamma_{\mathcal{A}}[\mathcal{A}_{q,n}^k]) \leq \mathrm{wcc}(\Gamma_{\mathcal{A}}[\mathcal{A}_{q,n}^{\ell}])$.

(3) *If $S^{\mathcal{P}}$ is transitive on $\mathcal{P}_{q,n}^{\ell}$, then it is transitive on $\mathcal{P}_{q,n}^k$.*

Our work towards Theorems 1.1 and 1.5 makes use of the existing techniques developed in the corresponding work on permutation groups. To prepare a proof for Theorems 1.1 and 1.5 and to discuss relevant problems, we move on in Section 2 to a general setting in which the sets are assigned a valued poset structure and the transformations on the sets are assumed to be structure-preserving. We regard Theorem 1.1 as a statement on Boolean semirings, a special kind of valued posets, and so, following the general strategy described in Section 2, we can give a proof of Theorem 1.1 in Section 3. In Section 4, we initiate a discussion of the homogeneity problems for transformation semigroups acting on matroids, which provides much further direction to go beyond Theorem 1.1. Especially, we sketch a proof of Theorem 1.5 there.

2 Valued posets

This section looks technical but our subject here is to formulate an approach of getting results like Theorems 1.1 and 1.5, which is used in different concrete settings in the literature on permutation groups already.

For ease of notation, for any two sets Ω and Ψ, if they are different or if we do not emphasize that they may be equal, the image of $\omega \in \Omega$ under a map $g \in \Psi^{\Omega}$ is denoted $g(\omega)$; recall that it is often written as ωg in many places of the paper when $\Omega = \Psi$. A **poset** P consists of a set Ω and a binary relation $<_P$ on it which is transitive and acyclic;

namely we require that $\alpha <_P \alpha$ never happens, and that $\alpha <_P \beta$ and $\beta <_P \gamma$ implies $\alpha <_P \gamma$ for all $\alpha, \beta, \gamma \in \Omega$. We often just write P for its ground set Ω and we say the poset P is finite if $|P|$ is finite. For each $\alpha \in P$, the **principal ideal** generated by α is the set $\{\beta : \beta <_P \alpha\} \cup \{\alpha\} \subseteq P$, which we denote by $P_\downarrow(\alpha)$; the **principal filter** generated by α is the set $\{\beta : \alpha <_P \beta\} \cup \{\alpha\} \subseteq P$, which we denote by $P_\uparrow(\alpha)$. A map g from a poset P to a poset Q is **order-preserving** if $g(\beta) \in Q_\downarrow(g(\alpha))$ holds whenever $\beta \in P_\downarrow(\alpha)$. We use $\mathrm{End}(P)$ to denote the set of all order-preserving maps from P to itself.

Let $\mathbb{Z}_{\geq 0}$ be the set of nonnegative integers which carries a natural poset structure such that $a < b$ in $\mathbb{Z}_{\geq 0}$ if and only if $b - a$ is a positive integer. A **valuation** on a poset P is a order-preserving map r_P from P to the poset $\mathbb{Z}_{\geq 0}$. When we say P is a **valued poset**, we are considering the poset P together with a valuation r_P, though the valuation may be only implicitly indicated. The **rank** of a valuated poset P, denoted by $\mathrm{r}(P)$, is the maximum value of $\mathrm{r}_P(\alpha)$ for $\alpha \in P$ if it exists and is ∞ otherwise. For a poset P, the symbols like $<_P$ and r_P will often be abbreviated to $<$ and r when no confusion can arise. Let P be a valued poset. For any $k \in \mathbb{Z}_{\geq 0}$, we write P_k for the set $\{\alpha \in P : \mathrm{r}(\alpha) = k\}$. For any nonnegative integers $k \leq \ell$, we call the poset P (k, ℓ)**-finite** provided $P_k \neq \emptyset$, $P_\ell \neq \emptyset$ and the set $P_\ell \cap P_\uparrow(\alpha)$ is finite for every $\alpha \in P_k$; we call P (ℓ, k)**-finite** provided $P_k \neq \emptyset$, $P_\ell \neq \emptyset$ and the set $P_\downarrow(\beta) \cap P_k$ is finite for every $\beta \in P_\ell$; we call $g \in \mathrm{End}(P)$ a (k, ℓ)**-hereditary endomorphism** if $\mathrm{r}_P(g(\alpha)) = \mathrm{r}_P(\alpha) = k$ ensures that g induces a bijection from the set $P_\ell \cap P_\uparrow(\alpha)$ to $P_\ell \cap P_\uparrow(\alpha g)$ for all $\alpha \in P_k$; we call $g \in \mathrm{End}(P)$ an (ℓ, k)**-hereditary endomorphism** if $\mathrm{r}_P(g(\alpha)) = \mathrm{r}_P(\alpha) = \ell$ ensures that g induces a bijection from the set $P_k \cap P_\downarrow(\alpha)$ to $P_k \cap P_\downarrow(\alpha g)$ for all $\alpha \in P_\ell$. For any $k, \ell \in \mathbb{Z}_{\geq 0}$, we designate by $\mathrm{hEnd}_{k,\ell}(P)$ the set of all (k, ℓ)-hereditary endomorphisms of the valued poset P.

Let S be a transformation semigroup on a valued poset P and let G be a generating set of S. For any two positive integers k and ℓ with $k < \ell \leq \mathrm{r}(P)$, we set $\Pi_{S,G}(k, \ell)$ to be the digraph with vertex set P_k and arc set $\{(\alpha, \alpha') : \exists g \in G, \beta \in P_\ell \text{ s.t. } \beta g \in P_\ell, \alpha \in P_k \cap P_\downarrow(\beta), \alpha' = \alpha g\}$; we set $\Pi_{S,G}(\ell, k)$ to be the digraph with vertex set P_k and arc set $\{(\alpha, \alpha') : \exists g \in G, \beta \in P_\ell \text{ s.t. } \alpha g \in P_\ell, \alpha \in P_k \cap P_\uparrow(\beta), \alpha' = \alpha g\}$. We use the shorthand $\Pi_S(k, \ell)$ for $\Pi_{S,S}(k, \ell)$.

Lemma 2.1. *Let P be a valued poset. Take two nonnegative integers k and ℓ satisfying $k, \ell \leq \mathrm{r}(P)$. Let S be a transformation semigroup acting on P such that $S \subseteq \mathrm{hEnd}_{\ell,k}(P)$, let G be a generator set of S, and let $\Gamma = \Gamma(S, G)$. Assume that P is (ℓ, k)-finite. If every weakly connected component of $\Gamma[P_\ell]$ is strongly connected, then so is $\Pi_{S,G}(k, \ell)$.*

Proof. Assume that $(\alpha, \alpha') \in \mathrm{E}(\Pi_{S,G}(k, \ell))$. Our task is to show that there is a walk from α' to α of positive length in $\Pi_{S,G}(k, \ell)$, namely there exists $f \in S$ such that $\alpha' f = \alpha$.

Without loss of generality, we assume that $k < \ell$. By the definition of $\Pi_{S,G}(k, \ell)$, we can find $g \in S$ and $\beta \in P_\ell$ such that $\alpha \in P_k \cap P_\downarrow(\beta)$, $\beta g \in P_\ell$ and $\alpha' = \alpha g$. Since every weakly connected component of $\Gamma[P_\ell]$ is strongly connected, we can find $h \in S$ such that $(\beta g)h = \beta$. Since $\beta(gh) = \beta$ and $gh \in \mathrm{hEnd}_{\ell,k}(P)$, it follows that gh induces a permutation on $P_k \cap P_\downarrow(\beta)$. But from the assumption that P is (ℓ, k)-finite, we see that $P_k \cap P_\downarrow(\beta)$ is a finite set. This means that there exists a positive integer r such that $\alpha(gh)^r = \alpha$. Accordingly, for $f = (hg)^{r-1}h$ it holds $\alpha' f = (\alpha g)(hg)^{r-1}h = \alpha(gh)^r = \alpha$, finishing the proof. □

For any set Ω, $\mathbb{Q}^\Omega$ refers to the linear space of all rational functions on Ω with finite supports. If P is a (k, ℓ)-finite valued poset, the **inclusion operator** $\zeta_P^{k,\ell} : \mathbb{Q}^{P_k} \to \mathbb{Q}^{P_\ell}$ is the linear operator such that for all $f \in \mathbb{Q}^{P_k}$ and $\beta \in P_\ell$

$$(\zeta_P^{k,\ell}(f))(\beta) = \begin{cases} \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha), & \text{if } k \leq \ell; \\ \sum_{\alpha \in P_k \cap P_\uparrow(\beta)} f(\alpha), & \text{if } k > \ell. \end{cases} \tag{2.1}$$

Lemma 2.2. *Let P be a valued poset and let k and ℓ be two nonnegative integers such that P_k and P_ℓ are both nonempty finite sets. Let S be a transformation semigroup on P such that $S \subseteq \mathrm{hEnd}_{\ell,k}(P)$ and let Γ stand for $\Gamma(S)$. Assume that $\zeta_P^{k,\ell}$ is an injective linear map from $\mathbb{Q}^{P_k}$ to $\mathbb{Q}^{P_\ell}$.*

(1) $\mathrm{wcc}(\Gamma[P_k]) \leq \mathrm{wcc}(\Pi_S(k,\ell)) \leq \mathrm{wcc}(\Gamma[P_\ell])$.

(2) *If $\Gamma[P_\ell]$ is strongly connected, then so is $\Gamma[P_k]$.*

Proof. (1) The first inequality is a consequence of the fact that $\mathrm{E}(\Pi_S(k,\ell)) \subseteq \mathrm{E}(\Gamma[P_k])$.

Let $W \subseteq \mathbb{Q}^{P_\ell}$ be the subspace of all functions which are constant on each weakly connected component of $\Gamma[P_\ell]$; let $V \subseteq \mathbb{Q}^{P_k}$ be the subspace of all functions which are constant on each weakly connected component of $\Pi_S(k,\ell)$. Note that $\dim(V) = \mathrm{wcc}(\Pi_S(k,\ell))$ and $\dim(W) = \mathrm{wcc}(\Gamma[P_\ell])$ and so it suffices to demonstrate $\dim(V) \leq \dim(W)$.

By symmetry, we only deal with the case of $k \leq \ell$. For every $f \in V$ and every arc $(\beta, \beta g)$ of $\Gamma[P_\ell]$, we have

$$\begin{aligned} (\zeta_P^{k,\ell}(f))(\beta g) &= \sum_{\alpha' \in P_k \cap P_\downarrow(\beta g)} f(\alpha') \\ &= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha g) \qquad (g \in \mathrm{hEnd}_{\ell,k}(P)) \\ &= \sum_{\alpha \in P_k \cap P_\downarrow(\beta)} f(\alpha) \qquad (f \in V) \\ &= (\zeta_P^{k,\ell}(f))(\beta). \end{aligned} \tag{2.2}$$

This says that $\zeta_P^{k,\ell}(V) \subseteq W$. Hence, by the injectivity of $\zeta_P^{k,\ell}$, $\dim(V) \leq \dim(W)$, as wanted.

(2) We know from (1) that $\mathrm{wcc}(\Pi_S(k,\ell)) = 1$. By Lemma 2.1, we further see that $\Pi_S(k,\ell)$ is strongly connected. Since $\Pi_S(k,\ell)$ is a spanning subgraph of $\Gamma[P_k]$, $\Gamma[P_k]$ must be strongly connected, finishing the proof. □

3 Boolean semirings

For any set Ω, the set $\mathrm{B}_\Omega \doteq \cup_{k=0}^{\infty} \binom{\Omega}{k}$ forms a poset under the inclusion relationship, which is often known as the **Boolean semiring over** Ω – the set 2^Ω gives rise to the Boolean algebra over Ω. When we view B_Ω as a valuated poset, unless stated otherwise, the valuation will be $\mathrm{r}(A) = |A|$ for all $A \in \mathrm{B}_\Omega$. For the valuated poset $P = \mathrm{B}_\Omega$ and $0 \leq k < \ell \leq |\Omega|$, we write the inclusion operator $\zeta_P^{k,\ell}$ defined in Eq. (2.1) as $\zeta_\Omega^{k,\ell}$. That is,

$$(\zeta_\Omega^{k,\ell}(f))(B) = \sum_{A \in \binom{B}{k}} f(A),$$

for all $f \in \mathbb{Q}^{\binom{\Omega}{k}}$ and $B \in \binom{\Omega}{\ell}$.

Following a common approach in establishing homogeneity of permutation groups [10, 25], we will make use of the ensuing classical result about inclusion matrices. For a simple proof of it, we refer the reader to [34, Theorem 2.4] and [13, Corollary].

Lemma 3.1. *Let Ω be a finite set. Let k and ℓ be two nonnegative integers such that $k \leq \ell \leq k+\ell \leq |\Omega|$. Then*

$$\ker \zeta_{\Omega}^{k,\ell} = \{0\}.$$

Let Ω be a set and S be a transformation semigroup on Ω. Let $\Omega^* \doteq \{(\omega, C) : \omega \in C \in 2^{\Omega}\}$ and, for each $g \in S$, let g^* be the transformation on Ω^* which sends (ω, C) to $(\omega g, C\overline{g})$ for all $(\omega, C) \in \Omega^*$. Let S^* stand for the transformation semigroup on Ω^* consisting of all elements g^* for $g \in S$. For all positive integers ℓ, we use the following notation:

$$\Omega_\ell^* \doteq \{(\omega, C) : \omega \in C \in \binom{\Omega}{\ell}\}$$

and

$$\Gamma_\ell^*(S) \doteq \Gamma(S^*)[\Omega_\ell^*].$$

Here is a result analogous to Lemma 2.1.

Lemma 3.2. *Let m be a positive integer and let S be an m-homogeneous transformation semigroup acting on a set Ω. Then the digraph $\Gamma_m^*(S)$ is symmetric. Especially, every weakly connected component of $\Gamma_m^*(S)$ is strongly connected.*

Proof. Take $(\omega, C) \in \Omega_m^*$ and $g \in S$ such that $|C\overline{g}| = m$. Our task is to show the existence of $h \in S$ such that $(\omega g, C\overline{g})h^* = (\omega, C)$. As S is m-homogeneous, we can find $f \in S$ such that $C\overline{gf} = (C\overline{g})\overline{f} = C$. Hence, the fact that $|C| = m < \infty$ allows us to obtain a positive integer r for which $(gf)^r|_C$ is the identity map on C. This means that we can choose h to be $f(gf)^{r-1}$. □

Lemma 3.3. *Let Ω be a set, let m be an integer satisfying $|\Omega| \geq m > 1$, and let S be a transformation semigroup on Ω. For every $X \in \binom{\Omega}{m}$, it holds*

$$\mathrm{scc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma(S_X)) \leq \mathrm{wcc}(\Gamma_m^*(S)) = \mathrm{scc}(\Gamma_m^*(S)). \tag{3.1}$$

Moreover, if S is m-homogeneous, then

$$\mathrm{scc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma(S_X)) = \mathrm{wcc}(\Gamma_m^*(S)) = \mathrm{scc}(\Gamma_m^*(S)). \tag{3.2}$$

Proof. It is trivial that $\mathrm{wcc}(\Gamma(S_X)) = \mathrm{scc}(\Gamma(S_X))$ and so we call each strongly/weakly connected component of $\Gamma(S_X)$ a component. By Lemma 3.2, every weakly connected component of $\Gamma_m^*(S)$ is strongly connected. Therefore, we obtain $\mathrm{wcc}(\Gamma_m^*(S)) = \mathrm{scc}(\Gamma_m^*(S))$ and we can call each strongly/weakly connected component of $\Gamma_m^*(S)$ simply a component. To prove Eq. (3.1), let us find an injective map ψ from the set of components of $\Gamma(S_X)$ to the set of components of $\Gamma_m^*(S)$.

For each $\gamma \in X$, let the component of $\Gamma_m^*(S)$ containing (γ, X) be $\psi'(\gamma)$. Take $(\gamma_1, \gamma_2) \in \mathrm{E}(\Gamma(S_X))$. We may assume that $\gamma_1 g = \gamma_2$ and $X\overline{g} = X$ for some $g \in S$. As $(\gamma_1, X)g^* = (\gamma_1 g, X\overline{g}) = (\gamma_2, X)$, we see that $\psi'(\gamma_1) = \psi'(\gamma_2)$. For each component C of $\Gamma(S_X)$, we can now choose any $\gamma \in C$ and get a well-defined map ψ

by setting $\psi(C) = \psi'(\gamma)$. For every component C^* of $\Gamma^*_m(S)$, let $\phi(C^*)$ be the set $\{\gamma \in X : (\gamma, X) \in C^*\}$. It is routine to check that $\phi\psi(C) = C$ for every component C of $\Gamma(S_X)$, proving that ψ is injective, as desired.

If S is m-homogeneous, for every component C^* of $\Gamma^*_m(S)$, we have $\phi(C^*) \neq \emptyset$ and so ϕ and ψ are inverses of each other. This proves Eq. (3.2). □

For any positive integers s, t and m, let $\mathrm{R}(s,t,m)$ denote the minimum integer $N \geq s$ such that for any set Υ with $|\Upsilon| \geq N$ and any partition of $\binom{\Upsilon}{s}$ into t equivalence classes one can always find one equivalence class which contains $\binom{A}{s}$ for some $A \in \binom{\Upsilon}{m}$. The existence of this number is guaranteed by Ramsey's Theorem [17, Chap. 2] [30].

Proof of Theorem 1.1. (1) If Ω is a finite set, the result follows from Lemma 2.2 (2) and Lemma 3.1. We now assume that Ω is an infinite set. Surely, it suffices to show that S is k-homogeneous under the assumption that it is $(k+1)$-homogeneous for a positive integer k. The proof below essentially follows the proof presented by Bercov and Hobby for [6, Corollary 1] and also the proof of Roy for [31, Theorem].

Fix an element $Y \in \binom{\Omega}{k+1}$. Since S is $(k+1)$-homogeneous, for every $X \in \binom{\Omega}{k}$ we can find $g_X \in S$ such that $X\overline{g_X} \in \binom{Y}{k}$ and $(X, X\overline{g_X}) \in \mathrm{E}(\Pi_S(k,k+1))$. We define a partition $\mathscr{P}$ of $\binom{\Omega}{k}$ into equivalence classes such that Z_1 and Z_2 are equivalent if and only if $Z_1\overline{g_{Z_1}} = Z_2\overline{g_{Z_2}}$. Thanks to Ramsey's Theorem, we know the existence of the finite number $\mathrm{R}(k, k+1, k+1)$. This means that there exist $W \in \binom{Y}{k}$ and $Y' \in \binom{\Omega}{k+1}$ such that $W'\overline{g_{W'}} = W$ for all $W' \in \binom{Y'}{k}$. Take any $X \in \binom{\Omega}{k}$. By virtue of the fact that S is $(k+1)$-homogeneous, we can find $h \in S$ such that $X\overline{h} \in \binom{Y'}{k}, X(\overline{hg_{X\overline{h}}}) = W$ and $(X, W) \in \mathrm{E}(\Pi_S(k, k+1))$. So far, what we see is that all elements of $\binom{\Omega}{k}$ can reach the vertex W in the spanning subdigraph $\Pi_S(k, k+1)$ of $\Gamma[\binom{\Omega}{k}]$ in one step. Applying Lemma 2.1 now then yields (1). Instead of utilizing Lemma 2.1, another way to see (1) is to further show the existence of $U \in \binom{\Omega}{k}$ such that U can reach all elements of $\binom{\Omega}{k}$ in $\Gamma[\binom{\Omega}{k}]$ in one step. This can be done similar to the above process of getting the existence of W. Since S is $(k+1)$-homogeneous, for every $X \in \binom{\Omega}{k}$ we can find $h_X \in S$ and $Y_X \in \binom{Y}{k}$ such that $Y_X\overline{h_X} = X$. We define a partition $\mathscr{P}$ of $\binom{\Omega}{k}$ into equivalence classes such that X and Z are equivalent if and only if $Y_X = Y_Z$. We can now continue with Ramsey's Theorem as above but we shall leave it to interested readers to fill in details.

(2) This is direct from Lemma 2.2 (1) and Lemma 3.1.

(3) Since S is $(\ell+1)$-homogenous, it follows from Lemma 3.3 that

$$\mathrm{wcc}(\Gamma(S_A)) = \mathrm{scc}(\Gamma(S_A)) \leq \mathrm{wcc}(\Gamma^*_k(S)) = \mathrm{scc}(\Gamma^*_k(S))$$

and

$$\mathrm{wcc}(\Gamma(S_B)) = \mathrm{scc}(\Gamma(S_B)) = \mathrm{wcc}(\Gamma^*_{\ell+1}(S)) = \mathrm{scc}(\Gamma^*_{\ell+1}(S)).$$

It then remains to prove $\mathrm{wcc}(\Gamma^*_{\ell+1}(S)) \geq \mathrm{wcc}(\Gamma^*_k(S))$.

We regard Ω^* as a valued poset by putting $\mathrm{r}((\alpha, X)) = |X|$ and requiring $(\alpha, X) < (\beta, Y)$ if and only if $\alpha = \beta \in \Omega$ and $X \subsetneq Y \subseteq \Omega$. Note that $S^* \subseteq \mathrm{hEnd}_{\ell+1,k}(\Omega^*)$. In view of Lemma 2.2 (1), it is sufficient to show that $\zeta^{k,\ell}_{\Omega^*}$ is injective.

For each nonnegative integer m and each $\alpha \in \Omega$, let $\Omega^*_{m,\alpha} \doteq \{(\alpha, A) : (\alpha, A) \in \Omega^*_m\}$. Corresponding to the partition $\Omega^*_k = \cup_{\alpha\in\Omega}\Omega^*_{k,\alpha}$ and $\Omega^*_{\ell+1} = \cup_{\beta\in\Omega}\Omega^*_{\ell+1,\beta}$, the $\Omega^*_k \times \Omega^*_{\ell+1}$ matrix $\zeta^{k,\ell+1}_{\Omega^*}$ is viewed as a partitioned matrix with blocks $\zeta_{\alpha,\beta}$, which are the submatrices with row index set $\Omega^*_{k,\alpha}$ and column index set $\Omega^*_{\ell+1,\alpha}$, where $\alpha, \beta \in \Omega$. Observe that

$$\zeta_{\alpha,\beta} = \begin{cases} \zeta^{k-1,\ell}_{\Omega\setminus\{\alpha\}}, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(k-1) + \ell \le |\Omega| - 1$, it follows from Lemma 3.1 that $\zeta_{\alpha,\alpha} = \zeta^{k-1,\ell}_{\Omega\setminus\{\alpha\}}$ is of full row rank for all $\alpha \in \Omega$. This implies that $\zeta^{k,\ell+1}_{\Omega^*}$ is an injective linear map, as desired. □

Remark 3.4. Let Ω be a set, which is not necessarily finite. Let k and ℓ be two nonnegative integers with $k \le \ell \le k + \ell \le |\Omega|$. For all $f \in \mathbb{Q}^{\binom{\Omega}{\ell}}$ and $A \in \binom{\Omega}{k}$, we put

$$(\zeta^{\ell,k}_{\Omega}(f))(A) = \sum_{A\subseteq B} f(B).$$

Making use of Lemma 3.1, it is easy to see that the linear transformation $\zeta^{\ell,k}_{\Omega} : \mathbb{Q}^{\binom{\Omega}{\ell}} \to \mathbb{Q}^{\binom{\Omega}{k}}$ is always a surjective map. Unfortunately, we do not see if this observation is helpful to get a counterpart of Theorem 1.1 (2) or Theorem 1.1 (3) when Ω is an infinite set.

For $f \in \Psi^{\Omega}$, we sometimes need to talk about $f(\omega)$ for $\omega \notin \Omega$. Following the practice of those mathematics with natural multivalued operations [5, 9, 39], we create a universal "don't care" symbol $\star \notin \Psi$ and will set $f(\omega) = \star$. We often regard $\star$ as all possible values in Ψ and so, whenever we have some addition operation $+$ on Ψ, we extend it to $\Psi \cup \{\star\}$ by setting $\star + \psi = \star$ for all $\psi \in \Psi \cup \{\star\}$.

For any $g \in \Omega^{\Omega}$ and $f \in \mathbb{Q}^{\binom{\Omega}{k}}$, write $fg^{\dagger,k}$ for the element in $(\star \cup \mathbb{Q})^{\binom{\Omega}{k}}$, where $\star$ stands for "don't care" and can be thought of as the whole set $\mathbb{Q}$, such that the following holds for all $A \in \binom{\Omega}{k}$:

$$fg^{\dagger,k}(A) = \begin{cases} f(A\overline{g}), & \text{if } A\overline{g} \in \binom{\Omega}{k}; \\ \star, & \text{if } A\overline{g} \notin \binom{\Omega}{k}. \end{cases}$$

For any $g \in \Omega^{\Omega}$, denote by $\operatorname{Fix} g^{\dagger,k}$ the set of $f \in \mathbb{Q}^{\binom{\Omega}{k}}$ for which

$$fg^{\dagger,k}(A) \in \{f(A), \star\}$$

holds for all $A \in \binom{\Omega}{k}$. If $1 < k \le \ell \le |\Omega|$, the reasoning in Eq. (2.2) leads to the commutative diagram in Fig. 1, which implies that all elements of $\operatorname{Fix} g^{\dagger,k}$ are mapped by $\zeta^{k,\ell}_{\Omega}$ to $\operatorname{Fix} g^{\dagger,\ell}$ for any $g \in \Omega^{\Omega}$.

4 Finite linear spaces and beyond

4.1 Homogeneity

When discussing transformation semigroups, we may often be more interested in those which preserve some structures, say simplicial maps for simplicial complexes, continuous maps for topological spaces, ordering preserving maps for posets, or adjacency-preserving

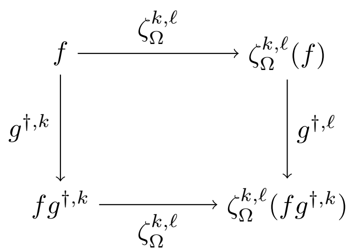

Figure 1: The inclusion operator intertwines with every transformation g.

maps in matrix geometry [32, 40]. Unlike the work on group actions on posets [3] and matroids [14], very little has been done on semigroup actions on these structures. We conclude the paper by addressing a bit those transformations which preserve "independence structure", namely morphisms among matroids.

A matroid M consists of a ground set $\mathscr{E}_M$ and a rank function r_M from $2^{\mathscr{E}_M}$ to the set of nonnegative integers plus infinity such that the rank axioms are satisfied [8, §1.5]. The flats of a matroid, ordered by inclusion, form a very pretty structure, called geometric lattice [16, p. 61]. Let M_1 and M_2 be two matroids and let f be a map from $\mathscr{E}_{M_1}$ to $\mathscr{E}_{M_2}$. We call f a **weak map from M_1 to M_2** provided

$$\mathrm{r}_{M_1}(A) \geq \mathrm{r}_{M_2}(A\overline{f})$$

holds for all $A \subseteq \mathscr{E}_{M_1}$ and $A\overline{f} = \{af : a \in A\} \subseteq \mathscr{E}_{M_2}$, and we call f a **strong map from M_1 to M_2** provided the preimage of any flat in M_2 is a flat of M_1 [21, 22, 35]. It is known that all strong maps must be weak maps.

Let M be a matroid on the ground set $E_M = \Omega$. Let $\mathrm{T}_M(\Omega)$ ($\mathrm{T}^*_M(\Omega)$) be the monoid consisting of all elements of $\mathrm{T}(\Omega)$ which are weak (strong) maps from M to itself. For each nonnegative integer t, let $\mathrm{F}_t(M)$ be the set of all rank-t flats of M. If we know that S is a subsemigroup of $\mathrm{T}_M(\Omega)$ ($\mathrm{T}^*_M(\Omega)$) acting on Ω, we can define a digraph $\Gamma_{M,t}(S)$ on $\mathrm{F}_t(M)$ as follows: for any $X, Y \in \mathrm{F}_t(M)$, there is an arc from X to Y if and only if there is $g \in S$ such that the minimum flat containing Xg in M is Y. What is the relationship between the connectivity of $\Gamma_{M,t}(S)$ and $\Gamma_{M,r}(S)$ for different t and r? We can ask the same question by imposing the extra condition that every element $f \in S$ is a bijection of Ω. If the matroid is a very special uniform matroid, namely a matroid in which all sets are independent, one can see that what is discussed in Section 1 becomes a very special case of this general setting. A result of Guiduli [4, Theorem 9.4] is more general than Lemma 3.1, which and its relatives should be useful for understanding the phase spaces of the semigroups of matroid morphisms.

Example 4.1. Let M be the non-Pappus matroid. The maximum rank of a flat of M is 3. Note that $|\,\mathrm{F}_1(M)| = 9$ and $|\,\mathrm{F}_2(M)| = 8$. If we take the trivial group $G = \{1\}$, then the number of its orbits on $\mathrm{F}_1(M)$ is larger than the number of its orbits on $\mathrm{F}_2(M)$.

Example 4.2. Let M be the Vámos matroid and let S be a subsemigroup of $T^*_M(\mathscr{E}_M)$. It holds $\mathrm{wcc}(\Gamma_{M,1}(S)) \leq \mathrm{wcc}(\Gamma_{M,2}(S)) \leq \mathrm{wcc}(\Gamma_{M,3}(S))$. Moreover, for the following three statements, we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) $\Gamma_{M,3}(S)$ is strongly connected;

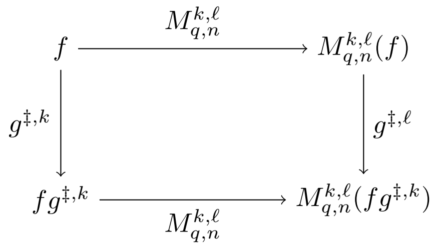

Figure 2: The inclusion operator intertwines with every linear transformation g.

(2) $\Gamma_{M,2}(S)$ is strongly connected;

(3) $\Gamma_{M,1}(S)$ is strongly connected.

Proof. Let P be the geometric lattice $\mathrm{F}(M)$. Let $f \in S$ and let $f' : P \to P$ be the map sending a flat X to the minimum flat containing $X\overline{f}$ in M. We can calculate that $\ker(\zeta_{\mathrm{F}(M)}^{k,\ell}) = \{0\}$ when $(k,\ell) \in \{(1,2),(1,3),(2,3)\}$. In light of Lemma 2.2, we will be done if we can show that $f' \in \mathrm{hEnd}_{\ell,k}(P)$ for $(k,\ell) \in \{(1,2),(1,3),(2,3)\}$, that is, we want to show that $f|_A$ induces a bijection on the set $P_k \cap P_\downarrow(A)$ for all $A \in P = \mathrm{F}(M)$ satisfying $\mathrm{r}_M(Af') = \mathrm{r}_M(A) = \ell$. If $f|_A$ is a bijection from A to Af', then it surely induces a bijection on the set $P_k \cap P_\downarrow(A)$. If $f|_A$ is not a bijection from A to Af', then we have $|A| = 4$ and $|Af'| = 3$. This implies the existence of $B \subseteq \binom{Af'}{2}$ such that $|f^{-1}(B)| = 3$. Note that $B \in \mathrm{F}(M)$ and $f^{-1}(B) \notin \mathrm{F}(M)$, which is impossible as f is assumed to be a strong map. □

Remark 4.3. Compared with the Fundamental Theorem of Projective (Affine) Geometry [12, 29], we think that weak/strong maps and bijective weak/strong maps for matroids are natural extensions of linear transformations and invertible linear transformations for linear spaces. We also mention the well-adopted viewpoint that full permutation group and the full transformation semigroup can be interpreted as the general linear group and the linear transformation semigroup over the field with one element. When the linear space is over a finite field, more results like Lemma 3.1 are known and so more progress can be expected.

As q-analogues of the set inclusion operator specified in Eq. (2.1), we define two linear transformations $M_{q,n}^{k,\ell} : \mathbb{Q}^{\mathcal{P}_{q,n}^k} \to \mathbb{Q}^{\mathcal{P}_{q,n}^\ell}$ and $N_{q,n}^{k,\ell} : \mathbb{Q}^{\mathcal{A}_{q,n}^k} \to \mathbb{Q}^{\mathcal{A}_{q,n}^\ell}$ as follows:

$$(M_{q,n}^{k,\ell}(f))(Y) \doteq \sum_{X \le Y, X \in \mathcal{P}_{q,n}^k} f(X),$$

and

$$(N_{q,n}^{k,\ell}(f'))(Y') \doteq \sum_{X' \le Y', X' \in \mathcal{A}_{q,n}^k} f(X'),$$

for all $f \in \mathbb{Q}^{\mathcal{P}_{q,n}^k}$, $Y \in \mathcal{P}_{q,n}^\ell$ and $f' \in \mathbb{Q}^{\mathcal{A}_{q,n}^k}$, $Y' \in \mathcal{A}_{q,n}^\ell$.

Take $g \in \mathrm{Mat}_n(\mathbb{F}_q)$, which naturally gives rise to a transformation on $\mathbb{F}_q^n$. For any $f \in \mathbb{F}_q^{\mathcal{P}_{q,n}^k}$, designate by $fg^{\ddagger,k}$ the element in $(\star \cup \mathbb{F}_q)^{\mathcal{P}_{q,n}^k}$, where $\star$ is the "don't care" symbol, such that the following holds for all $X \in \mathcal{P}^k_{q,n}$:

$$fg^{\ddagger,k}(X) = \begin{cases} f(X\overline{g}), & \text{if } X\overline{g} \in \mathcal{P}^k_{q,n}; \\ \star, & \text{if } X\overline{g} \notin \mathcal{P}^k_{q,n}. \end{cases}$$

Analogous to the deduction of Fig. 1, we can derive the commutativity of the diagram in Fig. 2 for all $f \in \mathbb{F}_q^{\mathcal{P}^k_{q,n}}$ on the condition that $1 < k \leq \ell \leq n$.

We are ready to present below a q-analogue of Lemma 3.1 and then those of Theorem 1.1 (1) and (2).

Lemma 4.4 (Kantor [19])**.** *Let k, ℓ and n be three nonnegative integers such that $k \leq \ell \leq k + \ell \leq n$ and let q be any prime power. Then*

$$\ker(M^{k,\ell}_{q,n}) = \{0\}$$

and

$$\ker(N^{k,\ell}_{q,n}) = \{0\}.$$

Proof of Theorem 1.5. Note that $S^{\mathcal{P}} \subseteq \text{hEnd}_{k,\ell}(\mathcal{P}_{q,n})$ and $S^{\mathcal{A}} \subseteq \text{hEnd}_{k,\ell}(\mathcal{A}_{q,n})$. The results are thus direct from Lemmas 2.2 and 4.4. □

Remark 4.5. Kantor [20, Theorem 2] determined all the ordered-basis-transitive finite geometric lattices of rank at least three: Roughly speaking, they are Boolean lattices, projective (affine) geometries, and four sporadic designs. Kantor's classification theorem along with Theorems 1.1 and 1.5 may be a basis for getting homogeneity results about ordered-basis-transitive matroids.

If we want to address linear spaces over infinite fields, we may need some other techniques. Indeed, we even do not know the answer to the small puzzle in the next example.

Example 4.6. Let n and k be two positive integers such that $k < n$. Fix a non-degenerate inner product on $\mathbb{Q}^n$, say $\langle , \rangle$. For each $g \in \text{GL}_n(\mathbb{Q})$, let $g^\top$ stand for the adjoint of g, namely the element such that $\langle ug, v\rangle = \langle u, vg^\top\rangle$ for all $u, v \in \mathbb{Q}^n$, and we write $g_\#$ for $(g^{-1})^\top$. Let $S \leq \text{GL}_n(\mathbb{Q})$ be a matrix group acting on $\mathbb{Q}^n$. If S is transitive on the set of all dimension-k subspaces and if $g_\# \in S$ for all $g \in S$, then S is transitive on the set of dimension-$(n-k)$ subspaces. To see this, fix a pair of subspaces (U, U') which are orthogonal complements to each other with respect to $\langle , \rangle$ and $(\dim U, \dim U') = (k, n-k)$. For each $g \in S$, we can see that $U\overline{g}$ and $U'\overline{g_\#}$ are orthogonal complements to each other with respect to the given inner product $\langle , \rangle$. Considering the set of pairs $\{(U\overline{g}, U'\overline{g_\#}) : g \in S\}$, we see that the transitivity on dimension-k subspaces implies transitivity on dimension-$(n-k)$ subspaces.

If we only know that S is a matrix semigroup and $(n, k) = (3, 2)$, can we still draw the conclusion that S is transitive on the set of dimension-$(n-k)$ subspaces from the assumption of its transitivity on dimension-k subspaces?

4.2 Duality: A result of Stanley

In mathematics we encounter quite some nice duality phenomena, say Chow's Theorem [28, Corollary 3.1] and many duality concepts for matroids [8]. This section aims to discuss the next result of Stanley [36, Corollary 9.9], for which we still do not know of a good counterpart for transformation semigroup.

Theorem 4.7 (Stanley). *Let F be a finite field and let k and n be two positive integers with $k < n$. For any subgroup G of $\mathrm{GL}(n, F)$, the number of orbits of the action of G on $\mathrm{Gr}(k, F^n)$ must be the same with the number of orbits of G acting on $\mathrm{Gr}(n-k, F^n)$.*

Let F be a field and Ω be a set. For each linear subspace $U \le F^\Omega$, let $U^\perp$ be the subspace of F^Ω given by

$$U^\perp \doteq \{f \in F^\Omega : \sum_{\omega\in\Omega} f(\omega)g(\omega) = 0 \text{ for all } g \in U\}.$$

Take a matrix $A \in F^{\Omega\times\Omega}$ and record its transpose by $A^\top$. For any $f \in F^\Omega$, which can be thought of as a row vector indexed by Ω, the image of f under the action of A, written as fA, can be thought of as the product of the row vector f and the matrix A. The matrix A induces a transformation $\widehat{A}$ on $\mathrm{Gr}(F^\Omega)$ such that $U \in \mathrm{Gr}(F^\Omega)$ is sent to $U\widehat{A} \doteq \{fA : f \in U\}$. It is easy to see that for any $U, W \in \mathrm{Gr}(V)$ we have the implication

$$U\widehat{A} = W \Longrightarrow W^\perp \widehat{A^\top} \le V^\perp; \tag{4.1}$$

Especially, when $A \in \mathrm{GL}_n(F)$ it holds

$$U\widehat{A} = W \Longleftrightarrow W^\perp \widehat{A^\top} = V^\perp. \tag{4.2}$$

According to Taussky and Zassenhaus [38], we can find $P \in \mathrm{GL}_n(F)$ such that $A^\top = PAP^{-1}$. This means that Eqs. (4.1) and (4.2) become

$$U\widehat{A} = W \Longrightarrow (W^\perp\widehat{P})\widehat{A} \le V^\perp\widehat{P}$$

and

$$U\widehat{A} = W \Longleftrightarrow (W^\perp\widehat{P})\widehat{A} = V^\perp\widehat{P}, \tag{4.3}$$

respectively. It is well-known that q-binomial coefficients (Gaussian coefficients) occur in pairs, namely in any n-dimensional linear space over a finite field, the number of k-dimensional subspaces is equal to the number of $(n-k)$-dimensional subspaces [16, Proposition 5.31] [37, §3]. This observation is the special case of Theorem 4.7 for G being the trivial group as well as a special case of Eq. (4.3) for A being the identity matrix. In general, as a consequence of Eq. (4.3), for any $A \in \mathrm{GL}_n(F)$, the number of k-dimension subspaces of F^n fixed by A equals to the number of $(n-k)$-dimension subspaces of F^n fixed by A. If F is a finite field and G is a subgroup of $\mathrm{GL}_n(F)$, in view of the Orbit Counting Lemma (also known as Burnside's Lemma), the above discussion leads to a proof of Theorem 4.7.

There is another way to verify Theorem 4.7. Let $G \le \mathrm{GL}_n(\mathbb{F}_q)$ and let k be a positive integer fulfilling $k \le \frac{n}{2}$. The group G can be seen as a permutation group acting on both $\mathrm{Gr}(n-k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^k$ and $\mathrm{Gr}(n-k, \mathbb{F}_q^n) = \mathcal{P}_{q,n}^{n-k}$; we use W_k and W_{n-k} for the two permutation modules accordingly. From Lemma 4.4 we see that $M_{q,n}^{k,n-k}$ is an $\mathbb{F}_q$-linear isomorphism from $\mathcal{P}_{q,n}^k$ to $\mathcal{P}_{q,n}^{n-k}$. Note that the commutative diagram in Fig. 2, which is the q-version of Fig. 1, will hold for all positive integers $k \le \frac{n}{2}$ (including k=1), if we assume furthermore that g comes from the group G. This then shows that W_k and W_{n-k} are isomorphic permutation modules for G. In particular, the number of orbits of G on $\mathcal{P}_{q,n}^k$ and the number of its orbits on $\mathcal{P}_{q,n}^{n-k}$ must be equal.

The above discussions are mainly about invertible linear operators over finite linear spaces. If we have a single linear operator $A \in \mathrm{Mat}_n(F)$, by considering its action on the linear space obtained by "collapsing" the kernel of A to zero, we can somehow still say something similarly as above. When we have a subsemigroup S of the full linear transformation monoid acting on a finite linear space, different elements of S may have different kernels and that makes it nontrivial to glean global information about the semigroup action. In general, if we have a transformation g on a set Ω, we get a partition of Ω in which two elements α and β fall into the same part provided $\alpha g = \beta g$, and we call this partition the kernel of g. For a permutation group, all elements of it have the same kernel. For a transformation semigroup, the existence of different kernels may make some arguments for permutation group invalid. We will study the kernels of elements from a transformation semigroup in [42].

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