BBT tree orders of countable connected graphs

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Abstract

Let G be a countable connected graph and let r be a vertex of G. We show that one can always find a spanning rooted tree T of G with root r so that for every edge xy of G which is outside of the edge set of T, it holds that x are y are incomparable in the tree order generated by the rooted tree T and that the father of x and the father of y in the rooted tree T are comparable in the same tree order. When G is finite, this result is established by Bonamy, Bousquet and Thomasse in their study of the Erdös-Hajnal conjecture. The countable version obtained here combined with our previous work on submodular functions leads to some consequence on paths with large closed neighbourhood in a graph weighted by a submodular function. We give examples to show that such a tree may not exist if we remove the countability assumption on G.

Keywords: good BBT subtree, infinite graph, rooted caterpillar, spanning tree.

1 Introduction

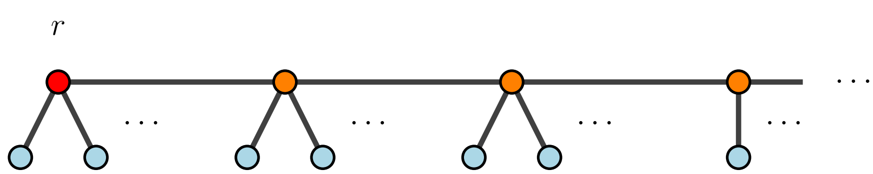

Figure 1: A rooted caterpillar.

A **graph** G consists of its vertex set $\mathrm{V}(G)$ and edge set $\mathrm{E}(G) \subseteq \binom{\mathrm{V}(G)}{2}$. For a graph G and a vertex $x \in \mathrm{V}(G)$, the neighbour of x in G, denoted by $\mathrm{N}_G(x)$, is the set $\{y : xy \in \mathrm{E}(G)\}$. More generally, for any $X \subseteq \mathrm{V}(G)$, let $\mathrm{N}_G(X)$ stand for the set $\bigcup_{x\in X} \mathrm{N}_G(x)$. We use the notation $\mathrm{N}_G[X]$ for $X \cup \mathrm{N}_G(X)$. For any $S \subseteq \mathrm{V}(G)$, $G[S]$ records the subgraph of G induced by S. A **rooted graph** is a graph G together with a distinguished vertex $\mathrm{r}_G \in \mathrm{V}(G)$, called the **root** of G. A **rooted tree** is a rooted graph in which each vertex has a unique path going to the root. A **rooted subgraph** of a graph G is a rooted graph H such that $\mathrm{r}_G = \mathrm{r}_H$ and H is a subgraph of G, namely $\mathrm{V}(H) \subseteq \mathrm{V}(G)$ and $\mathrm{E}(H) \subseteq \mathrm{E}(G)$. A rooted subgraph H of G is **spanning** if $\mathrm{V}(H) = \mathrm{V}(G)$. If we forget the specification of the root of a rooted graph, we return to its **underlying graph**. For any tree T and any two vertices $x, y \in \mathrm{V}(T)$, the set of vertices on the unique path between x and y in T is designated by $\overline{xTy}$. Let T be a rooted tree. Write $y \leq_T x$ if $x \in \overline{\mathrm{r}_T Ty}$. Note that $\leq_T$ is a partial order on $\mathrm{V}(T)$, known as the **tree-order** with respect to T. Define $\lceil x\rceil_T \doteq \{y : x \leq_T y\}$ and $\lfloor x\rfloor_T \doteq \{y : y \leq_T x\}$. The **children** of x in T is the set $\lfloor x\rfloor_T \cap \mathrm{N}_T(x)$, denoted by $\mathrm{C}_T(x)$; while the set of **descendants** of x in T is the set $\lfloor x\rfloor_T \setminus \{x\}$, denoted by $\mathrm{D}_T(x)$. If $x \neq \mathrm{r}_T$, the **father** of x in T is the unique element in $\lceil x\rceil_T \cap \mathrm{N}_T(x)$, denoted by $\mathrm{F}_T(x)$. A rooted tree T is called a **rooted caterpillar** if every two vertices of T of degree at least two must be comparable in the tree order $\leq_T$. For an illustration, see Fig. 1. The **central stalk** of a rooted caterpillar T is the union of r_T and the set of vertices of T having degree at least two. Denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively, the sets of reals, integers, and positive integers.

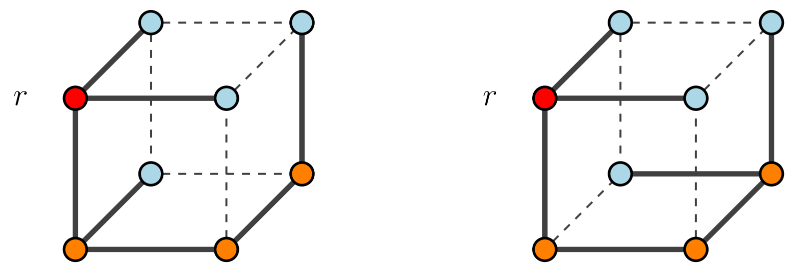

Figure 2: Two spanning rooted trees of the rooted 3-cube.

For infinite connected graphs, the existence of spanning rooted trees is equivalent to the Axiom of Choice or Zorn's Lemma [Sou08, §2.2]. Note that Zorn's Lemma is also known as Zorn's Maximal Theorem [Lei12] or Hausdorff Maximal Principle [Lew91]. A spanning rooted tree T of a rooted graph G is **normal** if the end vertices of every edge of G are comparable in the tree-order $\leq_T$. A normal spanning tree is also known as a depth-first search tree as the depth-first search on a finite connected graph will produce a normal spanning tree. Note that an uncountable complete graph obviously cannot have any normal spanning tree. There are vast research regarding which infinite connected graphs possess normal spanning trees [BK22, DL01, Hal00, LP99, Pit20, Pit21a, Pit21b]. More generally, people have been interested in understanding the shape of (rooted) spanning trees of infinite (di)graphs or infinite clique graphs [EPGJ$^+$21, HTL18, Joó18, Kom92, Pol91, Pol01, PGH21, RS22].

For any graph G, a rooted subtree T of it is called a **BBT subtree of** G if, for any $x, y \in \mathrm{V}(T)$ satisfying $xy \in \mathrm{E}(G) \setminus \mathrm{E}(T)$, it always holds that x and y are incomparable in $\leq_T$ but their fathers are comparable in $\leq_T$. Informally speaking, each non-tree edge of G contained in the vertex set of the BBT subtree T must "lean closely against" a chain in $\leq_T$.

Example 1. (1) *If there is $v \in \mathrm{V}(G)$ such that $\mathrm{N}_G[v] = \mathrm{V}(G)$, then a spanning BBT tree of G could be chosen as the one rooted at v obtained from G by deleting all edges not incident to v.*

(2) *Let G be the rooted graph with $\mathrm{V}(G) = \mathbb{R}$, $\mathrm{E}(G) = \{xy \in \binom{\mathbb{R}}{2} : \ |x - y| \leq 1\}$ and $\mathrm{r}_G = 0$. We can construct a spanning BBT tree T of G by specifying $\mathrm{E}(T)$ to be $E_0 \cup E_- \cup E_+$, where $E_0 = \{xy : \ x, y \in \mathbb{Z}, |x - y| = 1\}$, $E_- = \{xy : \ x \in \mathbb{R} \setminus \mathbb{Z}, x < 0, y = \lceil x \rceil\}$ and $E_+ = \{xy : \ x \in \mathbb{R} \setminus \mathbb{Z}, x > 0, y = \lfloor x \rfloor\}$.*

(3) *In Fig. 2, we paint two spanning trees of the rooted 3-cube, both of them are rooted caterpillars. Only the one on the left is a spanning BBT tree. Up to graph isomorphism, it is even the unique spanning BBT tree of the rooted 3-cube. We are wondering if there is any characterization of those countable connected rooted graphs whose spanning BBT trees are all isomorphic.*

(4) *We describe three spanning BBT trees of the rooted 4-cube in Fig. 3. Up to graph isomorphism, they are essentially all the spanning BBT trees in the rooted 4-cube. Note that the one on the right is not a rooted caterpillar.*

(5) *For the rooted n-cube, we write γ_n for its number of spanning BBT trees. One can check that $\gamma_1 = 1 = 1!$, $\gamma_2 = 2 = 2!$, $\gamma_3 = 6 = 3!$, $\gamma_4 = 72 = 4! \times 3$.*

Bonamy, Bousquet and Thomasse make use of spanning BBT trees in their study of the Erdös-Hajnal conjecture [BBT16, Theorem 6] while we appeal to it in our analysis of paths in pseudorandom graphs weighted by submodular functions [WZ22, Theorem 7]. Bonamy, Bousquet and Thomasse [BBT16, Lemma 2] observe that every finite connected graph has a spanning BBT tree. We find that if the given rooted graph possesses a well order on its vertex set which is of so-called finite type, then it has a spanning BBT tree [WZ22, Lemma 3]. Though the existence of a well order of finite type for any finite connected graph is trivial to see, we are even not clear about its existence for countable infinite graphs. Here is the main result of this note, whose proof, of course, needs the help of the axiom of countable choice.

Theorem 2. *Every connected countable rooted graph has a spanning BBT tree.*

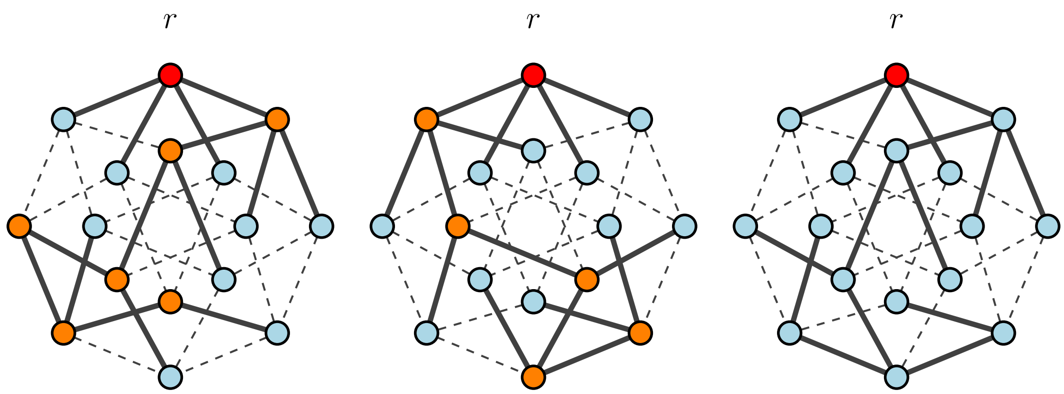

Figure 3: Three spanning BBT trees of the rooted 4-cube.

As indicated in [WZ22, Remark 1], Theorem 2 enables us to adapt the proof of [WZ22, Theorem 7] to derive the following result:

Corollary 3. *Let $k > 1$ be an integer and let $d_1, \ldots, d_k, c$ be $k+1$ positive reals. Let G be a countable connected graph and let μ be a submodular function defined on $2^{\mathrm{V}(G)}$ such that $\mu(\emptyset) \geq 0$ and*

$$\sup\{\mu(X) : \ |\mathrm{V}(G) \setminus X| \leq 1\} \geq \sum_{i=1}^{k} d_i + (k-1)c.$$

Then, either there exist k disjoint sets $D_1, \ldots, D_k \subseteq \mathrm{V}(G)$ such that $\mu(D_i) \geq d_i$ for all $i \in [k]$ of which no edge of G connects two different D_is, or there exist two subsets B and C of $\mathrm{V}(G)$ such that $C \subseteq B \subseteq \mathrm{N}_G[C]$, $G[C]$ is a path and $\mu(B) \geq c$.

For any $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, define $|x|_\infty = \max\{|x_1|, \ldots, |x_k|\}$ and $|x|_2 = \sqrt{x_1^2 + \cdots + x_k^2}$. Let us now present Theorem 4 as a dimension-two variant of Example 1 (2). If a rooted graph G has a rooted caterpillar T as its spanning rooted tree, to ensure that T is a spanning BBT tree of G, we only need to check that no edge of $\mathrm{E}(G) \setminus \mathrm{E}(T)$ will join a vertex at the central stalk of T to a vertex comparable with it in $\leq_T$; see Lemma 13. This explains why it is easier to construct a rooted caterpillar as a spanning BBT tree. [WZ22, Example 12] gives a spanning BBT tree of the infinite grid graph. The construction directly generalizes from $\mathbb{Z}^2$ to $\mathbb{R}^2$, thus leading to Theorem 4 (1).

Theorem 4. *In any of the following two cases, the rooted graph G has a rooted caterpillar as a spanning BBT tree.*

(1) *Let G be the rooted graph with $\mathrm{r}_G = (0,0)$, $\mathrm{V}(G) = \mathbb{R}^2$ and $\mathrm{E}(G) = \{xy \in \binom{\mathbb{R}^2}{2} : \ |x-y|_\infty \leq 1\}$.*

(2) *Let G be the rooted graph with $\mathrm{r}_G = (0,0)$, $\mathrm{V}(G) = \mathbb{R}^2$ and $\mathrm{E}(G) = \{xy \in \binom{\mathbb{R}^2}{2} : \ |x-y|_2 \leq 1\}$.*

Question 5. *Take an integer $k \geq 3$. For each of the following graphs G, we are wondering if it has any spanning BBT tree.*

(1) *Let G be the rooted graph with $\mathrm{V}(G) = \mathbb{R}^k$, $\mathrm{E}(G) = \{xy \in \binom{\mathbb{R}^k}{2} : \ |x-y|_\infty \leq 1\}$, and $\mathrm{r}_G = (0, \ldots, 0)$.*

(2) *Let G be the rooted graph with $\mathrm{V}(G) = \mathbb{R}^k$, $\mathrm{E}(G) = \{xy \in \binom{\mathbb{R}^k}{2} : \ |x-y|_2 \leq 1\}$, and $\mathrm{r}_G = (0, \ldots, 0)$.*

Theorem 6. *Let G be a graph. Assume that the following two conditions hold:*

(1) *For every countable set $X \subseteq \mathrm{V}(G)$, $\mathrm{N}_G[X] \neq \mathrm{V}(G)$.*

(2) *For every finite set $X \subseteq \mathrm{V}(G)$, $G[\mathrm{V}(G) \setminus \mathrm{N}_G[X]]$ is a connected graph.*

Then G has no spanning BBT tree.

Examples 7 and 8 illustrate that Theorem 2 does not extend to uncountable rooted graphs. Note that a similar difference between countable and uncountable also appears in unfriendly partitions of graphs [Has23].

Example 7. *Let G be a rooted graph with vertex set $\mathrm{V}(G) = \mathbb{R}^2$ in which two distinct vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if $(x_1 - y_1)(x_2 - y_2) = 0$. Take a countable set $X \subseteq \mathrm{V}(G)$. Note that the intersection of $\mathrm{N}_G[X]$ with $\{(x, x) : x \in \mathbb{R}\}$ is a countable set and so cannot be the whole $\mathrm{V}(G)$. In addition, for any two points $(x, y), (x', y') \in \mathbb{R}^2 \setminus \mathrm{N}_G[X]$, we find that they both belong to $\mathrm{N}_G[(x, y')]$ and $(x, y') \in \mathbb{R}^2 \setminus \mathrm{N}_G[X]$. We are thus ready to apply Theorem 6 and conclude that G has no spanning BBT tree.*

Example 8. *Let S be an uncountable set. Let G be a rooted graph with $\mathrm{V}(G) = \binom{S}{<\infty}$ and $\mathrm{E}(G) = \{AB : |A \triangle B| = 1\}$. Take a countable set $X \subseteq \mathrm{V}(G)$. Let $Y = \bigcup_{x \in X} x \subseteq S$, which is still a countable set. Take two distinct vertices $z_1, z_2 \in \mathrm{V}(G) \setminus Y$ and thus we see that $v \doteq \{z_1, z_2\} \in \mathrm{V}(G) \setminus \mathrm{N}_G[X]$. Denote by H the graph $G[\mathrm{V}(G) \setminus \mathrm{N}_G[X]]$. Take an arbitrary vertex $u \in \mathrm{V}(H)$, say $u = \{u_1, \ldots, u_n\}$. Let $s_1 = u \cup \{z_1\}$, $s_2 = u \cup \{z_1, z_2\}$ and $t_i = \{z_1, z_2\} \cup \{u_1, \ldots, u_{n-i}\}$ for $i \in [n]$. Observe that, for all $x \in X$ and $i \in [n]$, we have $|s_2 \triangle x| > |s_1 \triangle x| > |u \triangle x| \geq 2$ and $|t_i \triangle x| \geq |\{z_1, z_2\} \triangle x| = 2$. Hence $(u, s_1, s_2, t_1, \ldots, t_n = v)$ is a path in H. This means that H is a connected graph. An application of Theorem 6 now shows that G has no spanning BBT tree.*

Theorems 2 and 6 suggest the following question.

Question 9. *Let G and G' be two rooted graphs with the same underlying graph. Assume that G has a spanning BBT tree. Is it true that G' must have a spanning BBT tree as well?*

For those connected graphs without normal spanning trees, people have started to consider their "approximate" normal spanning trees [KMP21]. It may deserve to find counterpart of this line of research for spanning BBT trees.

In Section 2, we define good BBT subtrees of a countable rooted graph and introduce a partial order on good BBT subtrees; with the help of these concepts we finish a proof of Theorem 2 there. We play with rooted caterpillars and present a proof of Theorems 4 and 6 in Section 3.

2 Good BBT subtree

Let G be a connected rooted graph. A **good BBT subtree** of G is a quadruple $\mathcal{T} = (T, \lhd, \mathrm{c}, \mathrm{d})$, where T is a rooted subtree of G, $\lhd$ is an acyclic and transitive relation on $\mathrm{V}(\mathcal{T}) \doteq \mathrm{V}(T)$, and c, d are two maps from $\mathrm{V}(T)$ to $2^{\mathrm{V}(G)}$ such that the following hold:

T0 T is a BBT subtree of G.

T1 For any $\{x, y\} \in \binom{\mathrm{V}(T)}{2}$, x and y are comparable in $\lhd$ if and only if $\mathrm{F}_T(x) = \mathrm{F}_T(y)$.

T2 For each $x \in \mathrm{V}(T)$, the set $\mathrm{d}(x)$ coincides with the set of vertices y which are not equal to x but can reach x in the graph $G[\mathrm{R}_T(x)]$, where

$$\mathrm{R}_T(x) \doteq \begin{cases} \mathrm{V}(G), & \text{if } x = \mathrm{r}_T; \\ \Big(\mathrm{d}(\mathrm{F}_T(x)) \setminus \Big(\mathrm{c}(\mathrm{F}_T(x)) \cup \bigcup_{y \lhd x} \mathrm{d}(y)\Big)\Big) \cup \{x\}, & \text{otherwise.} \end{cases}$$

T3 For each $x \in \mathrm{V}(T)$, the set $\mathrm{c}(x)$ coincides with $\mathrm{N}_{G[\mathrm{d}(x)]}(x)$.

T4 For each $x \in \mathrm{V}(T)$, $\mathrm{C}_T(x) \subseteq \mathrm{c}(x)$ and $\mathrm{D}_T(x) \subseteq \mathrm{d}(x)$.

Note that a consequence of **T3** is that

$$\mathrm{d}(x) \subseteq \mathrm{R}_T(x) \subseteq \mathrm{d}(\mathrm{F}_T(x)) \tag{1}$$

for all $x \in \mathrm{V}(T) \setminus \{\mathrm{r}_T\}$.

Let G be a rooted graph. Let $\mathcal{T}_1 = (T_1, \lhd_1, \mathrm{c}_1, \mathrm{d}_1)$ and $\mathcal{T}_2 = (T_2, \lhd_2, \mathrm{c}_2, \mathrm{d}_2)$ be two good BBT subtrees of G. We say that $\mathcal{T}_2$ is an **extension** of $\mathcal{T}_1$ if T_1 is a rooted subtree of T_2 and $\lhd_1, \mathrm{c}_1, \mathrm{d}_1$ are the restrictions of $\lhd_2, \mathrm{c}_2, \mathrm{d}_2$ on $\mathrm{V}(T_1)$. In the case that $\mathcal{T}_1 = \mathcal{T}_2$, the extension is called a **trivial** extension.

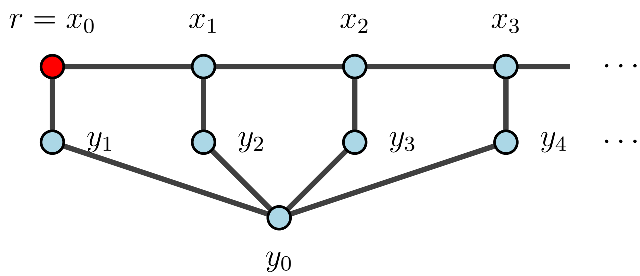

Figure 4: A rooted graph.

Roughly speaking, we want to utilize the concept of good BBT subtrees and their extensions to control the growing of BBT subtrees so that the growing process will not get stuck before arriving at a spanning tree; see the proof of Theorem 2. Given a good BBT subtree $\mathcal{T} = (T, \lhd, \mathrm{c}, \mathrm{d})$, for each vertex $x \in \mathrm{V}(\mathcal{T})$, the set $\mathrm{c}(x)$ is regarded as the potential children of x and the set $\mathrm{d}(x)$ is regarded as the potential descendent set of x for the future extensions of $\mathcal{T}$. When we let the existing good BBT subtree grow a new leaf x, we will demand $y \lhd x$ for all previous children y of its father; see the proof of Lemma 12.

Example 10 shows that a good BBT subtree of a connected graph may not have any nontrivial extension even if it is not a spanning tree. However, Lemma 12 claims that every non-spanning finite good BBT subtree of a connected rooted graph has a nontrivial extension, which is the workhorse for our proof of Theorem 2. Our proof of Lemma 12 will base on an additional technical lemma, Lemma 11.

Example 10. *Let G be the rooted graph with $\mathrm{V}(G) = \{x_{i-1}, y_{i-1} :\ i \in \mathbb{Z}_+\}$, $\mathrm{E}(G) = \{x_{i-1}x_i, x_{i-1}y_i, y_iy_0 :\ i \in \mathbb{Z}_+\}$ and $\mathrm{r}_G = x_0$; see Fig. 4. Then a spanning BBT tree H of G can be chosen by setting $\mathrm{E}(H) = \{y_iy_0, x_ix_{i+1} :\ i \in \mathbb{Z}_+\} \cup \{x_0y_1, y_2x_1\}$.*

Let T be the rooted subtree of G induced by $\mathrm{V}(G) \setminus \{y_0\}$. Let the binary relation $\lhd$ on $\mathrm{V}(T)$ be $\{(x_i, y_i) :\ i \in \mathbb{Z}_+\}$. Define c and d to be the maps from $\mathrm{V}(T)$ to $2^{\mathrm{V}(G)}$ such that for all $i \in \mathbb{Z}_+$, $\mathrm{c}(x_{i-1}) = \{x_i, y_i\}$, $\mathrm{c}(y_i) = \emptyset$, $\mathrm{d}(x_{i-1}) = \{x_j, y_j :\ j \geq i-1\} \cup \{z\}$ and $\mathrm{d}(y_i) = \{y_i\}$.

One can check that $\mathcal{T} = (T, \lhd, \mathrm{c}, \mathrm{d})$ is a good BBT subtree of G. Assume there exists a nontrivial extension $\mathcal{T}' = (T', \lhd', \mathrm{c}', \mathrm{d}')$ of $\mathcal{T}$. Then we have $y_0 \in \mathrm{V}(T')$. For $y_i = \mathrm{F}_{T'}(y_0)$, we now derive that $\mathrm{C}_{T'}(y_i) = \{y_0\} \nsubseteq \emptyset = \mathrm{c}(y_i) = \mathrm{c}'(y_i)$, which contradicts **T4**.

Lemma 11. *Let G be a connected rooted graph. Let $\mathcal{T} = (T, \lhd, \mathrm{c}, \mathrm{d})$ be a good BBT subtree of G. Suppose that T does not contain any infinite path. Take $v \in \mathrm{V}(G) \setminus \{\mathrm{r}_G\}$ and let $X_v \doteq \{x \in \mathrm{V}(T) :\ v \in \mathrm{d}(x)\}$. Then X_v contains a unique vertex x_0 such that $X_v = \lceil x_0 \rceil_T$.*

Proof. It clearly holds that $\mathrm{r}_T \in X_v$, and so X_v is nonempty. Note that for any $x \in X_v \setminus \{\mathrm{r}_T\}$, we have $v \in \mathrm{d}(x) \subseteq \mathrm{d}(\mathrm{F}_T(x))$. Since T does not contain any infinite path, it remains to demonstrate the fact that X_v does not contain two incomparable elements in $\leq_T$, which we prove by contradiction below.

If there were two elements from X_v which are incomparable in $\leq_T$, then we will find two distinct elements $x, y \in X_v$ satisfying $\mathrm{F}_T(x) = \mathrm{F}_T(y)$. Without loss of generality, assume that $x \lhd y$. It then follows from **T3** that

$$\mathrm{d}(y) \subseteq \mathrm{R}_T(y) \subseteq \mathrm{d}(\mathrm{F}_T(y)) \setminus \mathrm{d}(x).$$

Henceforth, we have $v \in \mathrm{d}(y) \cap \mathrm{d}(x) = \emptyset$, which is absurd. □

Lemma 12. *Let G be a connected rooted graph. Let $\mathcal{T} = (T, \lhd, \mathrm{c}, \mathrm{d})$ be a finite good BBT subtree of G. For every $v \in \mathrm{V}(G) \setminus \mathrm{V}(T)$, there exists an extension $\mathcal{T}'$ of $\mathcal{T}$ with $v \in \mathrm{V}(\mathcal{T}')$.*

Proof. According to Lemma 11, we may assume that $X_v = \lceil x_0 \rceil_T$. Let $P = (x_0, x_1, \ldots, x_s = v)$ be a shortest path between x_0 and v in $G[\mathrm{d}(x_0)]$. Let T' be the unique minimum rooted tree which has T as a rooted subgraph and has P as a subgraph. Define $\lhd'$ to be the acyclic and transitive relation on $\mathrm{V}(T')$ such that $y \lhd' x$ if and only if either $y \lhd x$ or $x = x_1, \mathrm{F}_{T'}(x) = \mathrm{F}_{T'}(y) = \mathrm{F}_T(y) = x_0$. Define c' and d' be the maps from $\mathrm{V}(T')$ to $2^{\mathrm{V}(T')}$ such that $\mathrm{c}'|_{\mathrm{V}(T)} = \mathrm{c}$, $\mathrm{d}'|_{\mathrm{V}(T)} = \mathrm{d}$ and $(T', \lhd', \mathrm{c}', \mathrm{d}')$ fulfills **T2** and **T3**.

For $i = 0, \ldots, s$, let $\mathcal{T}_i = (T_i, \lhd_i, \mathrm{c}_i, \mathrm{d}_i)$ represent the restriction of $\mathcal{T}'$ on the set $\mathrm{V}(T_i) \doteq \mathrm{V}(T) \cup \{x_0, \ldots, x_i\}$. Our task is to prove that $\mathcal{T}' = (T', \lhd', \mathrm{c}', \mathrm{d}')$ is a good BBT subtree of G. For this purpose, in view of Eq. (1) and Lemma 11, we only need to show that $\mathcal{T}_1$ is a good BBT subtree of G, that $\lceil x_1 \rceil_{T_1}$ is the set of all vertices x from $\mathrm{V}(T_1)$ satisfying $v \in \mathrm{d}_1(x)$, and that $\mathrm{d}_1(x_1) \supseteq \{x_2, \ldots, x_s\}$. By the same reasoning, one then obtains inductively that $\mathcal{T}_2, \ldots, \mathcal{T}_s = \mathcal{T}'$ are all good BBT subtrees of G, thus completing the proof.

It is clear from our construction that $\mathcal{T}_1$ satisfies **T1**, **T2** and **T3**.

We now verify **T4** for $\mathcal{T}_1$. A useful observation is

$$x_1 \in \mathrm{N}_{G[\mathrm{d}(x_0)]}(x_0) = \mathrm{c}(x_0) \subseteq \mathrm{d}(x_0). \tag{2}$$

Firstly, we show $\mathrm{C}_{T_1}(x) \subseteq \mathrm{c}_1(x)$ for all $x \in \mathrm{V}(T_1)$. For $x \in \mathrm{V}(T) \setminus \{x_0\}$, $\mathrm{C}_{T_1}(x) = \mathrm{C}_T(x) \subseteq \mathrm{c}(x) = \mathrm{c}_1(x)$; it is trivial that $\mathrm{C}_{T_1}(x_1) = \emptyset \subseteq \mathrm{c}_1(x_1)$; by Eq. (2) we have $\mathrm{C}_{T_1}(x_0) = \mathrm{C}_T(x_0) \cup \{x_1\} \subseteq \mathrm{c}(x_0) = \mathrm{c}_1(x_0)$.

Secondly, let us check that $\mathrm{D}_{T_1}(x) \subseteq \mathrm{d}_1(x)$ for all $x \in \mathrm{V}(T_1)$. For all $x \in \mathrm{V}(T) \setminus \lceil x_0 \rceil_T$, it holds $\mathrm{D}_{T_1}(x) = \mathrm{D}_T(x) \subseteq \mathrm{d}(x) = \mathrm{d}_1(x)$; it is clear that $\mathrm{D}_{T_1}(x_1) = \emptyset \subseteq \mathrm{d}_1(x_1)$; it follows from Eqs. (1) and (2) that $\mathrm{D}_{T_1}(x) = \mathrm{D}_T(x) \cup \{x_1\} \subseteq \mathrm{d}(x) \cup \mathrm{d}(x_0) = \mathrm{d}(x) = \mathrm{d}_1(x)$ for all $x \in \lceil x_0 \rceil_T$.

Finally, we have to verify **T0** for $\mathcal{T}_1$. Since T is a BBT subtree of G, we only need to prove for all $y \in (\mathrm{N}_G(x_1) \cap \mathrm{V}(T)) \setminus \{x_0\}$ that $y \notin \lceil x_0 \rceil_T$ and that $\mathrm{F}_{T_1}(y) = \mathrm{F}_T(y)$ and $\mathrm{F}_{T_1}(x_1) = x_0$ are comparable in $\le_{T_1}$.

By way of contradiction, let us assume that $y \in \lceil x_0 \rceil_T$. Let y' be the unique vertex from $\mathrm{N}_G(y) \cap \overline{yTx_0}$. By Eq. (1), it holds $x \in \mathrm{d}(x_0) \subseteq \mathrm{d}(y') \subseteq \mathrm{d}(y)$. Applying **T2** for $\mathcal{T}$ then yields $x \in \mathrm{c}(y)$ and so $x \notin \mathrm{d}(y')$, which is absurd.

After knowing $y \notin \lceil x_0 \rceil_T$, we intend to further demonstrate that $\mathrm{F}_T(y)$ and $\mathrm{F}_{T_1}(x_1)$ are comparable in $\le_{T_1}$. Let z be the smallest upper bound of y and x_0 in $\le_T$. Let $\{x'\} = \mathrm{N}_G(z) \cap \overline{zT_1x_1}$ and $\{y'\} = \mathrm{N}_G(z) \cap \overline{zTy}$. It suffices to show either $y = y'$ or $z = x_0$. Assuming $z \neq x_0$, there are two cases to consider.

Case 1. $x' \lhd_1 y'$.

If $y \neq y'$, then $y \in \mathrm{R}_T(y') \setminus \{y'\} \subseteq \mathrm{R}_T(x')$. In the graph $G[\mathrm{R}_T(x')]$, we can walk in T_1 from x' to x_1, then pass through the edge x_1y, and then walk in T from y to y'. This implies $y \in \mathrm{d}(x')$. But we also know that $y \in \mathrm{d}(y')$, which is impossible according to Lemma 11.

Case 2. $y' \lhd_1 x'$.

We indeed shall conclude that this case never happens. Since $z \neq x_0$, we have $x_1 \neq x'$. It thus follows $x_1 \in \mathrm{R}_{T'}(x') \setminus \{x'\} \subseteq \mathrm{R}_{T'}(y')$. In the graph $G[\mathrm{R}_{T'}(y')]$, we can reach y' from x_1 by passing through the edge x_1y and then walking along T to go from y to y'. This shows that $x_1 \in \mathrm{d}(y')$, which combined with the fact that $x_1 \in \mathrm{d}(x')$ surely violates the claim of Lemma 11. ☐

Proof of Theorem 2. Let G be a countable connected rooted graph. List all elements of $\mathrm{V}(G)$ as $\{x_i : i \in \mathbb{Z}_+\}$ so that $x_1 = \mathrm{r}_G$. Let $\mathcal{T}_1$ be the good BBT subtree of G consisting of the only vertex x_1. For each positive integer $i \ge 2$, we iteratively define a good BBT subtree $\mathcal{T}_i$ of G in the sequel. If $x_i \in \mathcal{T}_{i-1}$, then define $\mathcal{T}_i \doteq \mathcal{T}_{i-1}$; if $x_i \notin \mathcal{T}_{i-1}$, let $\mathcal{T}_i$ be an arbitrary extension of $\mathcal{T}$ satisfying $x_i \in \mathrm{V}(\mathcal{T}_i)$, whose existence is guaranteed by Lemma 12. We write T_i for the underlying rooted tree of $\mathcal{T}_i$ for all $i \in \mathbb{Z}_+$. Lastly, let T be the rooted subgraph of G with $\mathrm{V}(T) = \bigcup_{i \in \mathbb{Z}_+} \mathrm{V}(T_i) = \mathrm{V}(G)$ and $\mathrm{E}(T) = \bigcup_{i \in \mathbb{Z}_+} \mathrm{E}(T_i)$. It is clear that this T is a required spanning BBT tree of G. ☐

3 Uncountable graph

Lemma 13. *Let G be a rooted graph. Let $X \subseteq \mathrm{V}(G)$ be a set such that $G[X]$ is a path with r_G as one endpoint and that $\mathrm{N}_G[X] = \mathrm{V}(G)$. Then the rooted graph G has a rooted caterpillar as a spanning BBT tree.*

Proof. Let the finite or infinite path $G[X]$ be $(\mathrm{r}_G = x_0, x_1, \ldots)$. For each $v \in \mathrm{V}(G) \setminus X$, let $y_v \doteq x_i$ where i is the minimum nonnegative integer such that $x_i \in \mathrm{N}_G(v) \cap X$. Let T be the spanning rooted tree of G such that $\mathrm{E}(T) = \mathrm{E}(P) \cup \{vy_v : v \in \mathrm{V}(G) \setminus X\}$. It is easy to check that T is a rooted caterpillar and a spanning BBT tree of G. ☐

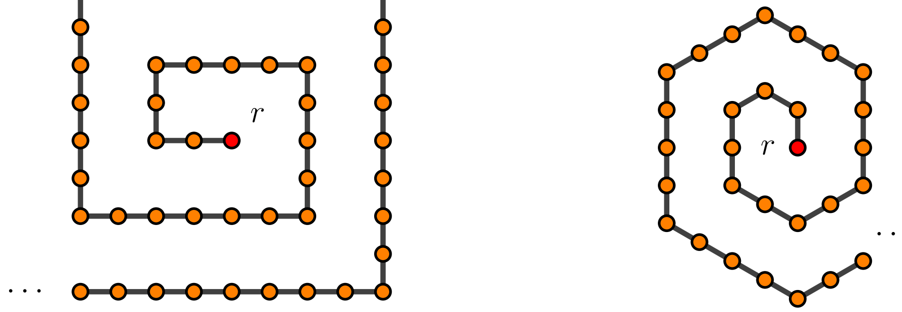

Figure 5: Central stalks of two rooted caterpillars on the plane.

Proof of Theorem 4. In view of Lemma 13, to specify a required spanning BBT tree T which is a rooted caterpillar, we need to find $X \subseteq \mathbb{R}^2$ so that $G[X]$ is a path and $\mathrm{N}_G[X] = \mathrm{V}(G)$; indeed, $G[X]$ is the central stalk of the rooted caterpillar as constructed in the proof of Lemma 13.

(1) Let $X = \bigcup_{k \in \mathbb{Z}_+} \{(y_1 - 1, 2 - 2k), (-2k, y_1 + 1), (y_2, 2k), (2k, y_2) :\ y_1, y_2 \in \mathbb{Z}, |y_1| \leq 2k - 1, |y_2| \leq 2k\}$. We have drawn $G[X]$ on the left of Fig. 5.

(2) Let $X = \bigcup_{k \in \mathbb{Z}_+} \{(2k-2)\alpha + (0, y-1), (0, 2k-1) + (y-2)\alpha, -2k\alpha + (y-1)\beta, -2k\beta + (0, y), -2k\beta + y\alpha, (0, -2k) + y\beta :\ y \in \mathbb{Z}_+, y \leq 2k\}$, where $\alpha = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $\beta = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. The central stalk of the resulting rooted caterpillar is depicted on the right of Fig. 5. □

Lemma 14. *Let T be a BBT subtree of a graph G and let $W \subseteq \mathrm{V}(T)$ be a set of pairwise incomparable elements in the tree order $\leq_T$. Assume that $\{x, y\} \in \binom{W}{2}$ and take $u \in \mathrm{D}_T(x)$ and $v \in \mathrm{D}_T(y)$. If P is a path connecting u and v in G, then P must pass through a vertex of G outside of $\cup_{w \in W} \mathrm{D}_T(w)$.*

Proof. Let the path P be $(u = u_0, u_1, \ldots, u_\ell = v)$. Assume, for sake of contradiction, that $u_0, \ldots, u_\ell$ all fall into $\cup_{w \in W} \mathrm{D}_T(w)$. This means that $\mathrm{F}_T(u_0), \ldots, \mathrm{F}_T(u_\ell)$ all belong to $\cup_{w \in W} \lfloor w \rfloor_T$. As distinct elements from W are incomparable in $\leq_T$, we know that $\lfloor w \rfloor_T$, $w \in W$, are pairwise disjoint. Note that $\mathrm{F}_T(u_0) \in \lfloor x \rfloor_T$ and $\mathrm{F}_T(u_\ell) \in \lfloor y \rfloor_T$. We thus find that there exists a positive integer $t \leq \ell$ such that $\mathrm{F}_T(u_{t-1}) \in \lfloor w \rfloor_T$ and $\mathrm{F}_T(u_t) \in \lfloor w' \rfloor_T$ for two distinct elements $w, w' \in W$. Consequently, we find that $u_t u_{t-1} \in \mathrm{E}(G) \setminus \mathrm{E}(T)$ while $\mathrm{F}_T(u_{t-1})$ and $\mathrm{F}_T(u_t)$ are incomparable in $\leq_T$. This contradicts the fact that T is a BBT subtree of G. □

Lemma 15. *Let G be a graph. Assume that, for every finite set $X \subseteq \mathrm{V}(G)$, $G[\mathrm{V}(G) \setminus \mathrm{N}_G[X]]$ is a connected graph. If T is a spanning BBT tree of G, then T is a rooted caterpillar.*

Proof. Since $v \leq_T \mathrm{r}_T$ holds for every $v \in \mathrm{V}(T)$, we derive from the definition of BBT subtree that $\mathrm{N}_T(\mathrm{r}_T) = \mathrm{N}_G(\mathrm{r}_T)$. We claim that there is at most one vertex x from $\mathrm{N}_T(\mathrm{r}_T)$ such that $\mathrm{D}_T(x) \neq \emptyset$. If this were not true, we will find four different vertices x, y, u, v such that $x, y \in \mathrm{N}_T(\mathrm{r}_T)$, $u \in \mathrm{D}_T(x)$ and $v \in \mathrm{D}_T(y)$. By Lemma 14, there does not exist any path of G connecting u to v and lying inside $\bigcup_{w \in \mathrm{N}_G(\mathrm{r}_T)} \mathrm{D}_T(w) = \mathrm{V}(G) \setminus \mathrm{N}_G[\mathrm{r}_T]$. But this is impossible as our assumption gives that $\mathrm{V}(G) \setminus \mathrm{N}_G[\mathrm{r}_T]$ induces a connected subgraph of G.

If there is no vertex x from $\mathrm{N}_T(\mathrm{r}_T)$ such that $\mathrm{D}_T(x) \neq \emptyset$, then T is surely a rooted caterpillar. Otherwise, we suppose that x is the unique vertex from $\mathrm{N}_T(\mathrm{r}_T)$ such that $\mathrm{D}_T(x) \neq \emptyset$. Let U be the set $\mathrm{N}_G[\mathrm{r}_T] \setminus \{x\}$. Let G' and T' be the subgraph obtained from G and T, respectively, by removing all vertices from U. We choose x as the root to make T' a rooted tree. It is clear that T' is a spanning BBT tree of G'. Observe that $\mathrm{V}(G') \setminus \mathrm{N}_{G'}[x] = \mathrm{V}(G) \setminus \mathrm{N}_G[X]$ for $X = \{x, \mathrm{r}_T\}$. Therefore, the same argument as above shows that there is at most one vertex x' from $\mathrm{N}_{T'}(\mathrm{r}_{T'}) = \mathrm{C}_T(x')$ such that $\mathrm{D}_T(x') = \mathrm{D}_{T'}(x') \neq \emptyset$.

Continuing in this way, we can see that T is a rooted caterpillar, as desired. □

Proof of Theorem 6. If G has a spanning BBT tree, say T, Condition (2) along with Lemma 15 says that T must be a rooted caterpillar. Let S be the set of vertices of T having degree at least two and let $X = S \cup \{\mathrm{r}_T\}$. Then we find that $\mathrm{N}_G[X] \supseteq \mathrm{N}_T[X] = \mathrm{V}(T) = \mathrm{V}(G)$, violating Condition (1). □

Declarations This work is supported by the National Natural Science Foundation of China 11971305 and by the Fundamental Research Funds for the Central Universities. The authors have no conflicts of interest to declare that are relevant to the content of this article. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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