Competition Numbers and Phylogeny Numbers of Connected Graphs and Hypergraphs *

Yanzhen Xiong, Soesoe Zaw and Yinfeng Zhu
School of Mathematical Sciences, Shanghai Jiao Tong University
Shanghai 200240, China
xyz920805@sjtu.edu.cn, soesoezaw@sjtu.edu.cn, fengzi@sjtu.edu.cn

Abstract

Let D be a digraph. The competition graph of D is the graph having the same vertex set with D and having an edge joining two different vertices if and only if they have at least one common out-neighbor in D. The phylogeny graph of D is the competition graph of the digraph constructed from D by adding loops at all vertices. The competition/phylogeny number of a graph is the least number of vertices to be added to make the graph a competition/phylogeny graph of an acyclic digraph. In this paper, we show that for any integer k there is a connected graph such that its phylogeny number minus its competition number is greater than k. We get similar results for hypergraphs.

keywords: quotient digraph, competition graph, phylogeny graph.
MSC 2010: 05C35.

1 Introduction

For any positive integer n, let $[n]$ denote the set $\{1, \ldots, n\}$. For any set $\{u, v\}$ of two elements, we often use the shorthand uv. A *graph* G comprises a *vertex set* $\mathrm{V}(G)$ and an *edge set* $\mathrm{E}(G)$ such that $\mathrm{E}(G) \subseteq \binom{\mathrm{V}(G)}{2}$. For a nonnegative integer ℓ, a *walk* of length ℓ in G between vertices v_0 and v_ℓ of G is a sequence $v_0, \ldots, v_\ell$ of vertices of G such that $v_{i-1}v_i \in \mathrm{E}(G)$ for all $i \in [\ell]$. We denote the length of a walk W by $|W|$. A graph G is called *connected* if for each pair of vertices in G, there is a walk in G between them.

A *digraph* D consists of a *vertex set* $\mathrm{V}(D)$ and an *arc set* $\mathrm{A}(D)$ such that $\mathrm{A}(D) \subseteq \mathrm{V}(D) \times \mathrm{V}(D)$. A *loop* in D at vertex v of D is the arc $(v, v) \in \mathrm{A}(D)$. For a digraph D, let D° denote the digraph obtained from D by adding loops, namely, $\mathrm{V}(D^\circ) = \mathrm{V}(D)$ and $\mathrm{A}(D^\circ) = \mathrm{A}(D) \cup \{(v, v) :\ v \in \mathrm{V}(D)\}$. For a positive integer ℓ, we call a sequence $v_0, v_1, \ldots, v_\ell$ of $\ell + 1$ vertices of D a *cycle* of length ℓ in D if $v_0 = v_\ell$, $(v_{i-1}, v_i) \in \mathrm{A}(D)$ for each $i \in [\ell]$, and $v_i \neq v_j$ for all distinct $i, j \in [\ell]$. A digraph is said to be *acyclic* if there is no cycle in it.

For a graph G (resp. digraph D) and for a set $X \subseteq \mathrm{V}(G)$ $\big($resp. $X \subseteq \mathrm{V}(D)\big)$, the subgraph (resp. subdigraph) of G (resp. D) *induced* by X, denoted by $G[X]$ (resp. $D[X]$), is the graph (resp. digraph) consisting of the vertex set X and the edge set $\mathrm{E}(G) \cap \binom{X}{2}$ $\big($resp. arc set $\mathrm{A}(G) \cap (X \times X)\big)$. For two graphs G and H, we write $H \lhd G$ if $H = G[\mathrm{V}(H)]$.

Let D be a digraph and let u and v be two not necessarily distinct vertices of D. We say that u is an *in-neighbor* of v in D and v is an *out-neighbor* of u in D if $(u, v) \in \mathrm{A}(D)$. The set of in-neighbors of v in D, denoted by $\mathrm{N}^-_D(v)$, is called the *in-neighborhood* of v in D, while the set of out-neighbors of v in D, denoted by $\mathrm{N}^+_D(v)$, is called the *out-neighborhood* of v in D.

Let D be a digraph D. The *competition graph* of D [Coh68], denoted by $\mathcal{C}(D)$, is the graph satisfying

$$\mathrm{V}\left(\mathcal{C}(D)\right) = \mathrm{V}(D) \ \text{ and } \ \mathrm{E}\left(\mathcal{C}(D)\right) = \left\{uv :\ N^+_D(u) \cap N^+_D(v) \neq \emptyset\right\}.$$

The competition graph of D°, which is designated by $\mathcal{P}(D)$, is called the *phylogeny graph* of D [RS98]. In the study of graphical models, a moral graph [Lau96, §2.1.1], which is obtained from an acyclic digraph by

*Supported by STCSM (17690740800) and NSFC (11671258).

"marrying parents and deleting directions" [Lau96, §3.2.2], turns out to be the phylogeny graph of an acyclic digraph. The triangulation of moral graphs plays an active role in Bayesian networks [LS88, Pea86].

Let i and j be positive integers and let G be a graph. An (i,j) *digraph* [LCKS17] is a digraph whose in-degrees are bounded by i and out-degrees are bounded by j. The graph G is an (i,j) *competition* (resp. (i,j) *phylogeny*) graph [EK18] if there exists an acyclic (i,j) digraph D such that $G = \mathcal{C}(D)$ $\big($resp. $G = \mathcal{C}(D^\circ)\big)$. The (i,j) *competition* (resp. (i,j) *phylogeny*) number of G, denoted by $\kappa_{(i,j)}(G)$ $\big($resp. $\phi_{(i,j)}(G)\big)$, is the minimum value of $|\operatorname{V}(D)| - |\operatorname{V}(G)|$, where D runs through all the (i,j) digraphs such that $G \lhd \mathcal{C}(D)$ $\big($resp. $G \lhd \mathcal{P}(D)\big)$. In most species, a child has only two biological parents, and so it is interesting to study $\kappa_{(2,j)}(G)$ and $\phi_{(2,j)}(G)$. It is immediate that $\kappa_{(i_1,j_1)}(G) \leq \kappa_{(i_2,j_2)}(G)$ and $\phi_{(i_1,j_1)}(G) \leq \phi_{(i_2,j_2)}(G)$ whenever $i_1 \geq i_2$ and $j_1 \geq j_2$. Denote by $\kappa(G)$ and $\phi(G)$ the *competition number* and the *phylogeny number* of G, respectively, such that

$$\kappa(G) = \lim_{i,j\to\infty} \kappa_{(i,j)}(G) \quad \text{and} \quad \phi(G) = \lim_{i,j\to\infty} \phi_{(i,j)}(G).$$

We say that G is a *competition* (resp. *phylogeny*) *graph* if $\kappa(G) = 0$ $\big($resp. $\phi(G) = 0\big)$.

A *hypergraph* H comprises its vertex set $V(H) \neq \emptyset$ and its edge set $E(G) \subseteq \binom{\operatorname{V}(H)}{\geq 2}$. Let D be a digraph. The *competition hypergraph* of D [ST04], denoted by $\mathcal{CH}(D)$, is the hypergraph consisting of the vertex set $\operatorname{V}\big(\mathcal{CH}(D)\big) = \operatorname{V}(D)$ and the hyperedge set

$$\mathscr{E}\big(\mathcal{CH}(D)\big) = \{e \in \binom{\operatorname{V}(D)}{\geq 2} : \ \exists v \in \operatorname{V}(D) \text{ s.t. } e = \operatorname{N}_D^-(v)\}.$$

Denoted by $\mathcal{PH}(D)$ the *phylogeny hypergraph* of D [WXZ19] such that

$$\mathcal{PH}(D) = \mathcal{CH}(D^\circ).$$

The ST-*competition number* (resp. ST-*phylogeny number*) of a hypergraph H, denoted by $\kappa_{\text{ST}}(H)$ $\big($resp. $\phi_{\text{ST}}(H)\big)$, is the least cardinality of the set $\operatorname{V}(D) \setminus \operatorname{V}(H)$ where D runs through all acyclic digraphs satisfying $H \lhd \mathcal{CH}(D)$ $\big($resp. $H \lhd \mathcal{PH}(D)\big)$.

The problems of computing $\kappa(G)$ and $\phi(G)$ are proved to be NP-complete [Ops82, RS98]. To calculate the competition numbers and the phylogeny numbers for various graph classes has been one of the important problems in the research of competition/phylogeny graphs. There are many related works [Kim17, KPS12, KS08, Kuh13, Rob78, WL10, WXZ19]. Note that the simple "adding loops" operation is all the difference between defining competition numbers and phylogeny numbers. To which extent can we understand the consequence of this small twist? Wu, Xiong and Zaw observed the following relationship between ϕ and κ in 2018.

Theorem 1. *(Wu-Xiong-Zaw [WXZ19, Theorem 9])*

(i) There exists a graph G such that $\phi(G) - \kappa(G) + 1 = k$ if and only if k is a nonnegative integer.

(ii) There exists a hypergraph H such that $\phi_{ST}(H) - \kappa_{ST}(H) + 1 = k$ if and only if k is a nonnegative integer.

Problem 2. *(i) Is it true that for every nonnegative integer k, there exists a connected graph G satisfying $\phi(G) - \kappa(G) + 1 = k$?*

(ii) Is it true that for every nonnegative integer k, there exists a connected hypergraph H satisfying $\phi_{ST}(H) - \kappa_{ST}(H) + 1 = k$?

(iii) For any nonnegative integer k, does there exist a graph G satisfying $\phi_{(2,j)}(G) - \kappa_{(2,j)}(G) + 1 = k$?

(iv) For any nonnegative integer k, does there exist a connected graph G satisfying $\phi_{(2,j)}(G) - \kappa_{(2,j)}(G) + 1 = k$?

(v) Let n be a positive integer. Put $a(n) = \max\{\phi(G) - \kappa(G) + 1 : \ |\operatorname{V}(G)| = n\}$. Can we determine $a(n)$? What is the asymptotic behavior of $a(n)$?

(vi) Can we classify the graphs G such that $\phi(G)-\kappa(G)+1=0$? More generally, can we classify the graphs G with a "small" value of $\phi(G)-\kappa(G)+1$?

Theorem 1 (i) is proved by calculating ϕ and κ for the disjoint union of some complete tripartite graphs. If we take a connected graph G from those graphs whose competition numbers and phylogeny numbers are known, say line graphs, chordal graphs, or complete tripartite graphs, we always find that the value of $\phi(G)-\kappa(G)+1$ vanishes. It is somehow surprising for us to finally discover that applying $\phi-\kappa+1$ to connected graphs can yield arbitrarily large values. The main result of this paper is the following, which and its proof may help to further tackle Problem 2.

Theorem 3. *(i) For any integer k, there exists a connected graph G such that $\phi_{(3,5)}(G)-\kappa_{(3,5)}(G)+1\geq k$.*

(ii) For any integer k, there exists a connected graph G such that $\phi(G)-\kappa(G)+1\geq k$.

(iii) For any integer k, there exists a connected hypergraph H such that $\phi_{\mathrm{ST}}(G)-\kappa_{\mathrm{ST}}(G)+1\geq k$.

To be able to get precise value of the competition number and phylogeny number of a connected graph, it is natural that we have to assume some good structural properties of the graph. But so far those graphs G with large difference between $\phi(G)$ and $\kappa(G)$ seem to be outside of those good graph classes which allow us to get exact values of $\phi(G)$ and $\kappa(G)$. Accordingly, though we will need some prelimary results from [KS08, WXZ19], our proof of Theorem 3 needs new constructions and has new perspectives which is not seen in the proof of Theorem 1.

There are more variants of the constructions of competition graphs, say double competition graphs [Sco87] and niche hypergraphs [GST16]. Accordingly, there arises the problem of calculating the corresponding parameters, say the double competition numbers [JLRS87] and the niche numbers [FG92]. For these constructions, problems like Problem 2 may also deserve some attention as any work in that line will demonstrate our success in analyzing the possibly intricate effect caused by the simple operation of "adding loops."

2 Proof

Let X and Y be two sets. The *disjoint union* of X and Y is the set $X\sqcup Y:=\{(x,1):\ x\in X\}\cup\{(y,2):\ y\in Y\}$. The *disjoint union* of two graphs G_1 and G_2, denoted by $G_1\sqcup G_2$, is the graph consisting of the vertex set $\mathrm{V}(G_1)\sqcup\mathrm{V}(G_2)$ and the edge set $\big\{(u,i)(v,i):\ uv\in\mathrm{E}(G_1)\cup\mathrm{E}(G_2),\ i\in[2]\big\}$. Similarly, the *disjoint union* of two digraphs D_1 and D_2, denoted by $D_1\sqcup D_2$, is the digraph with the vertex set $\mathrm{V}(D_1)\sqcup\mathrm{V}(D_2)$ and the arc set $\big\{\big((u,i),(v,i)\big):\ (u,v)\in\mathrm{A}(D_1)\cup\mathrm{A}(D_2),\ i\in[2]\big\}$. For any nonnegative integer n, we use $\mathrm{I}_n(G)$ to stand for the graph G together with additional n isolated vertices. Namely, $\mathrm{I}_n(G)=G\sqcup I_n$, where I_n is the graph with n vertices and no edges.

Remark 4. *It is easy to see that the competition number of G is the least nonnegative integer k such that $\mathrm{I}_k(G)=\mathcal{C}(D)$ for some acyclic digraph D.*

Let D be a digraph. We call $v\in\mathrm{V}(D)$ a *source vertex of D* if $\mathrm{N}^-_D(v)=\emptyset$ and we call v a *sink* or *terminal* vertex of D if $\mathrm{N}^+_D(v)=\emptyset$. We use $\mathrm{S}(D)$ and $\mathrm{T}(D)$ to denote the collection of source vertices and sink vertices of D, respectively. Let $\sim$ be an equivalence relation on $\mathrm{V}(D)$. We use the notation $[v]$ to represent the equivalence class containing v. Then the *quotient digraph $D/\sim$* is defined as follows:

$$\mathrm{V}(D/\sim)=\{[v]:\ v\in\mathrm{V}(D)\}\quad\text{and}\quad\mathrm{A}(D/\sim)=\{([u],[v]):\ uv\in\mathrm{A}(D)\}.$$

Let D_1 and D_2 be two digraphs, let $X\subseteq\mathrm{T}(D_1)$ and $Y\subseteq\mathrm{S}(D_2)$ be two sets of equal size, and let f be a bijection from X to Y. Define $\sim_f$ to be the equivalence relation on $\mathrm{V}(D_1)\sqcup\mathrm{V}(D_2)$ such that the vertices $(v,1)$ and $(u,2)$ are equivalent if and only if $f(v)=u$. We write $D_1\sqcup_f D_2$ for the digraph $D_1\sqcup D_2/\sim_f$.

Remark 5. *For each $\ell\in[2]$, let D_ℓ be an acyclic (i_ℓ,j_ℓ) digraph. For two sets $X\subseteq\mathrm{T}(D_1)$ and $Y\subseteq\mathrm{S}(D_2)$ of the same size and for a bijection $f\in Y^X$, it is easy to check that $D_1\sqcup_f D_2$ is an acyclic $(\max\{i_1,i_2\},\max\{j_1,j_2\})$ digraph and that $\mathrm{I}_{|X|}\big(\mathcal{C}(D_1\sqcup_f D_2)\big)=\mathcal{C}(D_1)\sqcup\mathcal{C}(D_2)$.*

Lemma 6. *Let G be a graph and let D be an acyclic (i,j) digraph satisfying $\mathrm{I}_k(G) = \mathcal{C}(D)$ for some nonnegative integer k. Let $\ell = \min\{k, |\,\mathrm{S}(D)|\}$ and let m be a positive integer. Then it holds*

$$\kappa_{(i,j)}(\bigsqcup_{m} G) \leq mk - \ell(m-1). \tag{1}$$

Proof. Let $D_1 = D_2 = \cdots = D_m = D$. For each $t \in [m]$, we choose a set $X_t \subseteq \mathrm{V}(D_t) \setminus \mathrm{V}(G) \subseteq \mathrm{T}(D_t)$ and a set $Y_t \subseteq \mathrm{S}(D_t)$ such that $|X_t| = |Y_t| = \ell$. Let $f_i \in {Y_{t+1}}^{X_i}$ be a bijection for each $t \in [m-1]$ and let $\bar{D}_t := D_t \sqcup_{f_t} \cdots \sqcup_{f_{m-1}} D_m$ for each $t \in [m]$. It follows from Remark 5 that $\bar{D}_1$ is an acyclic (i,j) digraph and that $\mathrm{I}_{mk}\left(\bigsqcup_m G\right) = \mathcal{C}(D_1) \sqcup \cdots \sqcup \mathcal{C}(D_m) = \mathrm{I}_\ell\left(\mathcal{C}(D_1) \sqcup \cdots \sqcup \mathcal{C}(D_{m-2}) \sqcup \mathcal{C}(\bar{D}_{m-1})\right) = \cdots = \mathrm{I}_{(m-1)\ell}\left(\mathcal{C}(\bar{D}_1)\right)$, implying Eq. (1), as wanted. $\square$

Lemma 7. *(Roberts-Sheng [RS98, Lemma 6]) For any two graphs G_1 and G_2, it holds $\phi(G_1 \sqcup G_2) = \phi(G_1) + \phi(G_2)$.*

Lemma 8. *Let G be a graph and let i and j be integers such that $\phi(G) = \phi_{(i,j)}(G)$. Then $\phi_{(i,j)}(\bigsqcup_{t=1}^m G) = \phi(\bigsqcup_{t=1}^m G) = m\,\phi(G)$ for every nonnegative integer m.*

Proof. Let D be an acyclic (i,j) digraph such that $G \lhd \mathcal{P}(D)$ and $|\,\mathrm{V}(D)| - |\,\mathrm{V}(G)| = \phi(G)$. Observe that $\bigsqcup_{t=1}^m G \lhd \mathcal{P}(\bigsqcup_{t=1}^m D)$ and $|\,\mathrm{V}(\bigsqcup_{t=1}^m D)| - |\,\mathrm{V}(\bigsqcup_{t=1}^m G)| = m\,\phi(G)$. This indicates that $\phi_{(i,j)}(\bigsqcup_{t=1}^m G) \leq m\,\phi(G)$. By Lemma 7, we then obtain $m\,\phi(G) = \phi(\bigsqcup_{t=1}^m G) \leq \phi_{(i,j)}(\bigsqcup_{t=1}^m G) \leq m\,\phi(G)$. This proves the lemma. $\square$

Let G be a graph. We define $\mathrm{Dist}_G(u,v)$ to be

$$\mathrm{Dist}_G(u,v) := \min\left\{|W| :\ W \text{ is a walk in } G \text{ between } u \text{ and } v\right\},$$

where we use the convention that the minimum of an empty set is $+\infty$. For two different vertices u and v of G, we write $G + uv$ for the graph consisting of the vertex set $\mathrm{V}(G)$ and the edge set $\mathrm{E}(G) \cup \{uv\}$. For any positive integer n, a *complete graph* of order n is a graph on n vertices and with all possible $\binom{n}{2}$ edges. A nonempty subset X of $\mathrm{V}(G)$ is a *clique* of G provided $G[X]$ is a complete graph. A clique of G is called *maximal* whenever it is not properly contained in another clique of G. Let D be a digraph. For two vertices u and v of D, we denote $D - (u,v)$ for the digraph with the vertex set $\mathrm{V}(D)$ and the arc set $\mathrm{A}(D) \setminus \{(u,v)\}$.

Lemma 9. *Let G be a graph and let u and v be two vertices of G such that $\mathrm{Dist}_G(u,v) \geq 3$. Then $\phi(G+uv) \geq \phi(G)$.*

Proof. Write $p = \phi(G + uv)$. To finish the proof, it suffices to construct a digraph D satisfying

$$|\,\mathrm{V}(D)| \leq |\,\mathrm{V}(G)| + p \tag{2}$$

and

$$G \lhd \mathcal{P}(D). \tag{3}$$

By the definition of phylogeny number, there exists an acyclic digraph D' such that $G + uv \lhd \mathcal{P}(D')$ and $|\,\mathrm{V}(D')| - |\,\mathrm{V}(G+uv)| = p$. Let $D = D'\left[\mathrm{V}(D')\right] - (u,v) - (v,u)$. It is easy to see $|\,\mathrm{V}(D)| = |\,\mathrm{V}(D')| = |\,\mathrm{V}(G+uv)| + p = |\,\mathrm{V}(G)| + p$. This proves Eq. (2).

To verify Eq. (3), we need to show both $\mathrm{E}\left(\mathcal{P}(D)\right) \cap \binom{\mathrm{V}(G)}{2} \subseteq \mathrm{E}(G)$ and $\mathrm{E}(G) \subseteq \mathrm{E}\left(\mathcal{P}(D)\right)$. To show $\mathrm{E}\left(\mathcal{P}(D)\right) \cap \binom{\mathrm{V}(G)}{2} \subseteq \mathrm{E}(G)$, it is suffice to prove $uv \notin \mathrm{E}\left(\mathcal{P}(D)\right)$. Since $\mathrm{Dist}_G(u,v) \geq 3$, uv is a maximal clique in $G + uv$. Then $N^+_{D'}(u) \cap N^+_{D'}(v) = \emptyset$. Therefore $uv \notin \mathrm{E}\left(\mathcal{P}(D' - (u,v) - (v,u))\right) = \mathrm{E}\left(\mathcal{P}(D)\right)$. We now try to prove $\mathrm{E}(G) \subseteq \mathrm{E}\left(\mathcal{P}(D)\right)$. Let $xy \in \mathrm{E}(G)$. Since $uv \notin \mathrm{E}(G)$, $|\{x,y\} \cap \{u,v\}| \in \{0,1\}$. We only need to consider the following two cases.

Case 1 $|\{x,y\} \cap \{u,v\}| = 0$.

By $\{x,y\} \cap \{u,v\} = \emptyset$, it holds that $N^+_D(x) = N^+_{D'}(x)$ and $N^+_D(y) = N^+_{D'}(y)$. Since $xy \in \mathrm{E}\left(\mathcal{P}(D')\right)$, we have $xy \in \mathrm{E}\left(\mathcal{P}(D)\right)$.

Case 2 $|\{x,y\} \cap \{u,v\}| = 1$.

Without loss of generality, we assume that $x = u$ and $y \neq v$. Since uv is a maximal clique in $G+uv$, we have $v \notin N^+_{D'}(u) \cap N^+_{D'}(y) = N^+_{D'}(x) \cap N^+_{D'}(y)$. By the construction, $N^+_D(y) = N^+_{D'}(y)$ and $N^+_D(x) = N^+_{D'}(x) \setminus \{v\}$. Then $xy \in \mathrm{E}\left(\mathcal{P}(D)\right)$. □

Let n and m be positive integers. We use the notation K^n_m for the *uniform complete multipartite graph* with m parts and uniform part size n, namely a graph whose vertex set is partitioned into n parts of equal size m such that two vertices are adjacent if and only if they are from different parts. Note that K^1_m is a complete graph of order m.

As a final preparation for our proof of Theorem 3, we recall two results from [KS08, WXZ19]. We mention that the claims on degree-bounded competition/phylogeny number in Lemmas 10 and 11 should be read from the proofs presented in [KS08, WXZ19].

Lemma 10. *(Kim-Sano [KS08, Theorem 1]) For $n \geq 2$, $\kappa(K^n_3) = n^2 - 3n + 4$. Moreover, there is an acyclic $(3,3)$ digraph D such that $\mathcal{C}(D) = \mathrm{I}_{n^2-3n+4}(K^n_3)$ and $|\mathrm{S}(D)| = n^2 - 3n + 4$, and so $\kappa_{(3,3)}(K^n_3) = n^2 - 3n + 4$.*

Lemma 11. *(Wu-Xiong-Zaw [WXZ19, Theorem 2(iii)]) For $n \geq 2$, $\phi(K^n_3) = n^2 - 3n + 3$ and $\phi_{(3,3)}(K^n_3) = n^2 - 3n + 3$.*

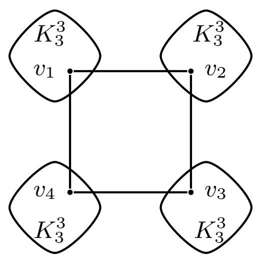

Figure 1: $\mathcal{G}_4$.

Proof of Theorem 3. Pick a positive integer m with $m \geq 4$. For each $t \in [m]$, let $G_t = K^3_3$ and let v_i be a vertex of G_i. Denote by $\mathcal{G}_m$ the connected graph

$$\bigsqcup_{t=1}^{m} G_t + v_1v_2 + v_2v_3 + \cdots + v_{m-1}v_m + v_mv_1.$$

See Fig. 1 for an illustration of $\mathcal{G}_m$ when $m = 4$.

To prove (i) and (ii), we assume, without loss of generality, that $k \geq 4$. By Lemmas 6 and 10 we obtain that $\kappa_{(3,3)}(\bigsqcup_{t=1}^{k} G_i) \leq 4$. According to the definition of competition graphs, it is easy to verify that

$$\kappa(\mathcal{G}_k) \leq \kappa_{(3,5)}(\mathcal{G}_k) \leq \kappa_{(3,3)}(\bigsqcup_{t=1}^{k} G_t) + |\,\mathrm{E}(\mathcal{G}_k) \setminus \mathrm{E}(\bigsqcup_{t=1}^{k} G_t)| = \kappa_{(3,3)}(\bigsqcup_{t=1}^{k} G_t) + k \leq 4 + k. \tag{4}$$

Since $k \geq 4$, no edge from $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$ can appear in a triangle of $\mathcal{G}_k$. Then we can apply Lemmas 8, 9 and 11 to obtain that

$$\phi_{(3,5)}(\mathcal{G}_k) \geq \phi(\mathcal{G}_k) \geq \phi(\bigsqcup_{t=1}^{k} G_t) = \phi_{(3,3)}(\bigsqcup_{t=1}^{k} G_t) = k\,\phi_{(3,3)}(K^3_3) = 3k. \tag{5}$$

Combining Eqs. (4) and (5), we know that $\mathcal{G}_k$ is a connected graph satisfying

$$\phi_{(3,5)}(\mathcal{G}_k) - \kappa_{(3,5)}(\mathcal{G}_k) + 1 \geq 3k - (k+4) + 1 = 2k - 3 > k,$$

and

$$\phi(\mathcal{G}_k) - \kappa(\mathcal{G}_k) + 1 \geq \phi(\mathcal{G}_k) - \kappa_{(3,5)}(\mathcal{G}_k) + 1 \geq 3k - (k+4) + 1 = 2k - 3 > k.$$

This is the proof of (i) and (ii).

We proceed to prove (iii). By Theorem 1 (ii), there exists a hypergraph H such that $\phi_{\mathrm{ST}}(H)-\kappa_{\mathrm{ST}}(H)+1 = k+1 \geq 0$. Surely, we only need to consider the case that $\mathrm{V}(H) \notin \mathscr{E}(H)$. Let H' be the connected hypergraph obtained by adding the hyperedge $\mathrm{V}(H)$ to H. Let D' be an acyclic digraph with $H' \lhd \mathcal{PH}(D')$. Then there exists a vertex v of D' such that $N^-_{D'}(v) \cup \{v\} = \mathrm{V}(H)$. Let D be the digraph obtained from D' by deleting the arcs in the set $\{(x, v) : x \in N^-_D(v)\}$. Note that $H \lhd \mathcal{PH}(D)$ and D is acyclic. This shows $\phi_{\mathrm{ST}}(H') \geq \phi_{\mathrm{ST}}(H)$. From the construction of H', it follows $\kappa_{\mathrm{ST}}(H') \leq \kappa_{\mathrm{ST}}(H) + 1$. Hence

$$\phi_{\mathrm{ST}}(H') - \kappa_{\mathrm{ST}}(H') + 1 \geq \phi_{\mathrm{ST}}(H) - (\kappa_{\mathrm{ST}}(H) + 1) + 1 = \phi_{\mathrm{ST}}(H) - \kappa_{\mathrm{ST}}(H) = k.$$

This proves (iii), as required. □

References

[Coh68] J. E. Cohen. *Interval Graphs and Food Webs: A finding and a problem.* RAND Corporation, 1968.

[EK18] S. Eoh and S.-R. Kim. On chordal phylogeny graphs. *Eprint Arxiv*, 2018. URL: arXiv:1810.11982.

[FG92] P. C. Fishburn and W. V. Gehrlein. Niche numbers. *Journal of Graph Theory*, 16(2):131–139, 1992. doi:10.1002/jgt.3190160204.

[GST16] C. Garske, M. Sonntag, and H.-M. Teichert. Niche hypergraphs. *Discussiones Mathematicae. Graph Theory*, 36(4):819–832, 2016. doi:10.7151/dmgt.1893.

[JLRS87] K. F. Jones, J. R. Lundgren, F. S. Roberts, and S. Seager. Some remarks on the double competition number of a graph. *Congressus Numerantium*, 60:17–24, 1987.

[Kim17] K. Kim. The competition numbers of Johnson graphs with diameter four. *Electronic Journal of Graph Theory and Applications*, 5(2):241–246, 2017. doi:10.5614/ejgta.2017.5.2.7.

[KPS12] S.-R. Kim, B. Park, and Y. Sano. The competition numbers of complete multipartite graphs with many partite sets. *Discrete Applied Mathematics*, 160(7-8):1176–1182, 2012. doi:10.1016/j.dam.2011.12.017.

[KS08] S.-R. Kim and Y. Sano. The competition numbers of complete tripartite graphs. *Discrete Applied Mathematics*, 156(18):3522–3524, 2008. doi:10.1016/j.dam.2008.04.009.

[Kuh13] J. Kuhl. Transversals and competition numbers of complete multipartite graphs. *Discrete Applied Mathematics*, 161(3):435–440, 2013. doi:10.1016/j.dam.2012.09.012.

[Lau96] S. L. Lauritzen. *Graphical Models*, volume 17 of *Oxford Statistical Science Series*. The Clarendon Press, Oxford University Press, New York, 1996. URL: http://www.oupcanada.com/catalog/9780198522195.html.

[LCKS17] S. C. Lee, J. Choi, S.-R. Kim, and Y. Sano. On the phylogeny graphs of degree-bounded digraphs. *Discrete Applied Mathematics*, 233:83–93, 2017. doi:10.1016/j.dam.2017.07.018.

[LS88] S. L. Lauritzen and D. J. Spiegelhalter. Local computations with probabilities on graphical structures and their application to expert systems. *Journal of the Royal Statistical Society. Series B. Methodological*, 50(2):157–224, 1988. URL: https://www.jstor.org/stable/2345762.

[Ops82] R. J. Opsut. On the computation of the competition number of a graph. *SIAM Journal on Algebraic and Discrete Methods*, 3(4):420–428, 1982. doi:10.1137/0603043.

[Pea86] J. Pearl. Fusion, propagation, and structuring in belief networks. *Artificial Intelligence*, 29(3):241–288, 1986. doi:10.1016/0004-3702(86)90072-X.

[Rob78] F. S. Roberts. Food webs, competition graphs, and the boxicity of ecological phase space. In *Theory and Applications of Graphs*, pages 477–490. Springer, Berlin, 1978.

[RS98] F. S. Roberts and L. Sheng. Phylogeny numbers. *Discrete Applied Mathematics*, 87(1-3):213–228, 1998. doi:10.1016/S0166-218X(98)00058-4.

[Sco87] D. D. Scott. The competition-common enemy graph of a digraph. *Discrete Applied Mathematics*, 17(3):269–280, 1987. doi:10.1016/0166-218X(87)90030-8.

[ST04] M. Sonntag and H.-M. Teichert. Competition hypergraphs. *Discrete Applied Mathematics*, 143(1-3):324–329, 2004. doi:10.1016/j.dam.2004.02.010.

[WL10] Y. Wu and J. Lu. Dimension-2 poset competition numbers and dimension-2 poset double competition numbers. *Discrete Applied Mathematics*, 158(6):706–717, 2010. doi:10.1016/j.dam.2009.12.001.

[WXZ19] Y. Wu, Y. Xiong, and S. Zaw. Competition numbers and phylogeny numbers: Uniform complete multipartite graphs. *Graphs and Combinatorics*, Mar 2019. doi:10.1007/s00373-019-02023-4.