Lecture Notes on Algebraic Combinatorics

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This is SJTU's graduate X071558, taught by Professor Eiichi Bannai. The formal name for this course is "代数组合 (Algebraic Combinatorics)". This set of notes is organized by 祝隐峰 (Yinfeng Zhu) as the T.A. of the course and is texified by 成燊 (Shen Cheng), 陈伟钊 (Weizhao Chen), 马振伟 (Zhenwei Ma), 荣石 (Shi Rong), 王周宁馨 (Zhouningxin Wang), 赵达 (Da Zhao) and 朱艳 (Yan Zhu). The textbook of the course is 代数的組合せ論入門 [BBI16].

Contents

1 Association Scheme: Definition and Examples

Definition 1.1: Association scheme

Let X be a finite set and $R_i \subset X \times X$ be binary relations for $i \in \{0, 1, \ldots, d\}$. $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ is called an **association scheme** of **class** d if the conditions (1) to (4) are satisfied. Moreover, an association scheme is **commutative** if the condition (5) satisfied. And an association scheme is **symmetric** if the condition (6) satisfied.

(1) $R_0 = \{(x, x) : x \in X\}$;

(2) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ where $R_i \cap R_j = \emptyset$ if $i \neq j$;

(3) For each $i \in \{0, 1, \ldots, d\}$, ${}^tR_i := \{(y, x) : (x, y) \in R_i\} = R_j$ for some $j \in \{0, 1, \ldots, d\}$. We use $R_{i'}$ to denote ${}^tR_i$;

(4) For each $i, j, k \in \{0, 1, \ldots, d\}$, the cardinality $|\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$ is a constant p_{ij}^k only depending on i, j, k whenever $(x, y) \in R_k$;

(5) $p_{ij}^k = p_{ji}^k$ for every $i, j, k \in \{0, 1, \ldots, d\}$;

(6) ${}^tR_i = R_i$ for each $i \in \{0, 1, \ldots, d\}$.

Remark 1.2

If $\mathfrak{X}$ is a symmetric association scheme, then $\mathfrak{X}$ is a commutative association scheme.

Proof. For every $(x, y) \in R_k$, we have

$$\begin{aligned} p_{ij}^k &= \big|\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}\big| \\ &= \big|\{z \in X : (z, x) \in R_i, (y, z) \in R_j\}\big| \\ &= p_{ji}^k. \end{aligned}$$

□

Now let's give some examples.

Example 1.3: Association scheme from finite group

Let G be a finite groups. Let G acts on a finite set Ω transitively. For $g \in G$, define $g : G \to G$ by $g(x) = x^g$ for each $x \in \Omega$. Then G acts on $\Omega \times \Omega$ by $(x, y)^g = (x^g, y^g)$. Let $R_0, R_1, \ldots, R_d$ be the orbits of G on $\Omega \times \Omega$. Then $\mathfrak{X} = (\Omega, \{R_i\}_{0 \leqslant i \leqslant d})$ is an association scheme.

Proof. Check the conditions (1) to (4) in Definition 1.1.

(1) By the transitivity, $R_0 = \{(x, x) : x \in \Omega\}$.

(2) By the definition of orbits, we have $(R_0, R_1, \ldots, R_d)$ form a partition of $\Omega \times \Omega$.

(3) By the definition of orbits, we can check ${}^tR_i$ is also an orbit, for all $i \in \{0, 1, \ldots, d\}$.

(4) Suppose $(x_1, y_1), (x_2, y_2) \in R_k$. Since R_k is a G-orbit, there exists $g \in G$ such that

$$(x_2, y_2) = (x_1, y_1)^g = (x_1^g, y_1^g).$$

It follows that for all $i, j \in \{0, 1, \ldots, d\}$,

$$z_1 \in \{z \in \Omega : (x_1, z) \in R_i, (z, y_1) \in R_j\}$$

if and only if

$$z_1^g \in \{z \in \Omega : (x_2, z) \in R_i, (z, y_2) \in R_j\}.$$

Hence p_{ij}^k's only depends on i, j, k but not on the choice of $(x, y) \in R_k$.

□

Exercise 1

Let D_{10} be the dihedral group of order 10. Note that D_{10} acts on the pentagon transitively. Let $\mathfrak{X}$ be the association scheme obtained from D_{10} in the way of Example 1.3. Calculate all p_{ij}^k's of the association scheme $\mathfrak{X}$.

Example 1.4: Hamming association scheme

Let F be a finite set with the cardinality $|F| = q \geqslant 2$. Define $X := F^d = F \times \cdots \times F$ with $|X| = q^d$. Let $x = (x_1, , \ldots, x_d)$ and $y = (y_1, , \ldots, y_d)$ be two point in X, the **Hamming distance** between x and y is defined to be the number of positions in which they differ, that is,

$$d(x, y) = \big|\{j : x_j \neq y_j\}\big|.$$

Define $R_i \subset X \times X$ to be

$$R_i = \{(x, y) \in X \times X : d(x, y) = i\}.$$

for all $i \in \{0, 1, \ldots, d\}$. Then $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ is an association scheme. It is called the **Hamming association scheme**, denoted by $H(d, q)$.

Proof. Let S_q be the symmetric group on F. Define S by $S := (S_q)^d = \overbrace{S_q \times \cdots \times S_q}^{d}$ with $|S| = (q!)^d$, that is,

$$S = \{\sigma = (\sigma_1, \ldots, \sigma_d) : \sigma_i \in S_q, i \in \{1, \ldots, d\}\}.$$

The group S acts on X by

$$x^\sigma = (x_1, \ldots, x_d)^\sigma = (x_1^{\sigma_1}, \ldots, x_d^{\sigma_d})$$

for $x \in X$ and $\sigma \in S$. Let S_d be the symmetric group on $\{1, \ldots, d\}$. The group S_d acts on X by

$$x^\tau = (x_1, \ldots, x_d)^\tau = (x_{1^{\tau_1}}, \ldots, x_{d^{\tau_d}})$$

for $x \in X$ and $\tau \in S_d$.

Note that S, S_d are the subgroups of the symmetric group on X, denoted by S_X. For each $x \in X$, $x^{\tau^{-1}\sigma\tau} = (x_1^{\sigma_{1^\tau}}, \ldots, x_d^{\sigma_{d^\tau}})$ and then $\tau^{-1}\sigma\tau = (\sigma_{1^\tau}, \ldots, \sigma_{d^\tau}) \in S$. Then S_d normalizes S. Hence SS_d is a subgroup of S_X on X. Actually, SS_d is usually called the **wreath product** $S_q \wr S_d$. The order of $S_q \wr S_d$ is $d!(q!)^d$. By the transitivity of S, we obtain the transitivity of $S_q \wr S_d$. We can check that $R_0, \ldots, R_d$ are all orbits of $S_q \wr S_d$. By Example 1.3, $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ becomes an association scheme. □

Note that Hamming association scheme is symmetric and then commutative. But calculating all the p_{ij}^ks of $H(d, q)$ is a bit complicated.

Exercise 2

(1) Calculate all the p_{ij}^k's of $H(2, 4)$.

(2) Calculate all the p_{ij}^k's of $H(2, q)$.

Example 1.5: Johnson association scheme

Let V be a finite set with the cardinality $|V| = v$. Let X be the set of all the d-elements subsets of V defined by

$$X := \binom{V}{d} = \{x \subset V : |x| = d\}.$$

Define $R_i = \{(x, y) \in X \times X : |x \cap y| = d - i\}$ for $i \in \{1, \ldots, d\}$. Then $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ is an association scheme. It is called the **Johnson association scheme**, denoted by $J(v, d)$.

Proof. The group $G = S_v$ acts on $X = \binom{V}{d}$ transitively. Then R_i's are the orbits of G on $X \times X$. It leads to $\mathfrak{X}$ an association scheme. □

Exercise 3

(1) Calculate all the p_{ij}^k's of $J(5, 2)$.

(2) Calculate all the p_{ij}^k's of $J(v, 2)$.

Example 1.6: Group association scheme

Let G be a finite group. Let $\{1\} = C_0, C_1, \ldots, C_d$ be all the conjugacy classes of G. For $i \in \{0, 1, \ldots, d\}$, define R_i by $R_i = \{(x, y) \in G \times G : y^{-1}x \in C_i\}$. Then $\mathfrak{X}(G) := (G, \{R_i\}_{0 \leqslant i \leqslant d})$ is a commutative association scheme. It is called the group association scheme of G.

Proof. The group $G \times G$ acts on G as follows: $x^{(g,h)} = g^{-1}xh$ for $(g, h) \in G \times G$ and $x \in G$. Then $G \times G$ acts on G transitively.

Claim

Each R_i is an orbit of $G \times G$ on $G \times G$ by the definition that $(x, y)^{(g,h)} = (x^{(g,h)}, y^{(g,h)}) = (g^{-1}xh, g^{-1}yh)$ for $(x, y), (g, h) \in G \times G$.

Proof of Claim. On the one hand, note that $(x,y) \in R_i$ is equivalent to $y^{-1}x \in C_i$. Since

$$(g^{-1}yh)^{-1}(g^{-1}xh) = h^{-1}y^{-1}gg^{-1}xh = h^{-1}y^{-1}xh \in C_i$$

for $y^{-1}x \in C_i$, we conclude that

$$(x,y)^{(g,h)} \in R_i$$

for each $(x,y) \in R_i$. Then each R_i is $G \times G$-invariant.

On the other hand, suppose $(x_1,y_1),(x_2,y_2) \in R_i$, then there exists $h \in G$, such that $h^{-1}(y_1^{-1}x_1)h = y_2^{-1}x_2$. It follows that $x_2h^{-1}(x_1^{-1}y_1)h = y_2$. Set $g = x_1hx_2^{-1}$.

$$\begin{aligned}(x_1,y_1)^{(g,h)} &= (g^{-1}xh, g^{-1}yh)\\ &= ((x_1hx_2^{-1})^{-1}x_1h, (x_1hx_2^{-1})^{-1}y_1h)\\ &= ((x_2h^{-1}x_1^{-1})x_1h, (x_2h^{-1}x_1^{-1})y_1h = (x_2,y_2).\end{aligned}$$

This shows that $G \times G$ acts on C_i transitively for all $i \in \{0,1,\ldots,d\}$ and completes the proof. □

By Example 1.3, $\mathfrak{X}(G) = (G, \{R_i\}_{0\leqslant i\leqslant d})$ becomes an association scheme. For commutativity, we show it as follows. For $(x,y) \in R_k$, note that

$$\begin{aligned}\{z \in G : (x,z) \in R_i, (z,y) \in R_j\} &= \{z \in G : z^{-1}x \in C_i, y^{-1}z \in C_j\}\\ &= xC_i^{-1} \cap yC_j = y(y^{-1}xC_i^{-1} \cap C_j).\end{aligned}$$

If we set $y^{-1}x = a \in C_k$, then $p_{ij}^k = |aC_i^{-1} \cap C_j|$. Note that $aC_ja^{-1} = C_j$, we have

$$p_{ij}^k = |aC_j^{-1} \cap C_i| = |C_ja^{-1} \cap C_i^{-1}| = |aC_ja^{-1} \cap aC_i^{-1}| = |aC_i^{-1} \cap C_j| = p_{ji}^k.$$

□

Question 1.1

Given a finite group G, in particular, a simple group. We have known that $\mathfrak{X}(G)$ is a commutative association scheme. We can also calculate all the p_{ij}^k's (this will discuss in the following lecture). Can $\mathfrak{X}(G)$ be shown to be a unique association scheme with the given parameters p_{ij}^k's of $\mathfrak{X}(G)$? If not, can you determine all the association schemes with the same p_{ij}^k's on $\mathfrak{X}(G)$?

There are several results.

- $G = A_5$. $\mathfrak{X}(G)$ is unique with the parameter p_{ij}^k's of $\mathfrak{X}(A_5)$ [Tom98].
- $G = \mathrm{PSL}(2,7)$. $|G| = 168$ and $\mathfrak{X}(\mathrm{PSL}(2,7))$ is unique with the parameter p_{ij}^k's of $\mathfrak{X}(\mathrm{PSL}(2,7))$[Tom99].

Problem is open for any other finite simple groups, such as A_n and $\mathrm{PSL}(2,\mathrm{q})$.

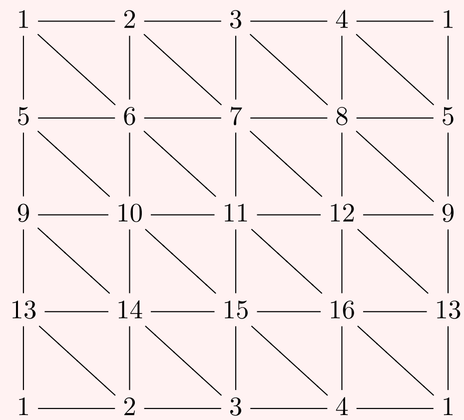
Is $G \to \mathfrak{X}(G)$ an one to one corresponding between the set of finite groups and the set of group association schemes. There is a counter example.

Example 1.7

Denote the dihedral group of order 8 by D_8 and the quaternion group of order 8 by Q_8. Then $\mathfrak{X}(D_8) \cong \mathfrak{X}(Q_8)$.

Can we find two different **simple** group G_1, G_2 with $\mathfrak{X}(G_1) \cong \mathfrak{X}(G_2)$? If we use the classification of finite simple groups, then one can prove that there are no such G_1 and G_2 by calculate the p_{ij}^k's of $\mathfrak{X}(G_1)$ and $\mathfrak{X}(G_2)$. But can we prove this directly, i.e., without using the classification of finite simple groups?

Example 1.8: Shrikhande graph



Let X be the set of vertices of the Shrikhande graph. Define $R_0 = \{(x,x) : x \in X\}$, $R_1 = \{(x,y) : x$ and y is joined by an edge $\}$ and $R_2 = \{(x,y) : x$ and y is not joined by an edge $\}$. Then $\mathfrak{X} = (X, \{R_0, R_1, R_2\})$ is an association scheme with 16 vertices.

But $\mathfrak{X}$ can not come from a finite group G acting on X. Otherwise, R_0, R_1, R_2 are all the orbits of a group G acting on $X \times X$. Since R_1 is an orbit, $(x,y) \in R_1$ after acted by $g \in G$ is still in R_1, that is, the acting of G keep the edges preserved. For instance, $(1,7),(1,11) \in R_2$ are in the same orbits, which satisfy $(1,7) = (1,11)^g$ for an existing g in G. Note that the common neighbors of $1,7$ are 2 and 6 while the common neighbors of $1,11$ are 6 and 16. We can never find an action to transform two isolated vertex pair $(6,16) \in R_2$ to an edge $(2,6) \in R_1$. Contradiction.

The graph defined on (X, R_1) of $H(2,4) = L_4$ has the same vertex set as the above $\mathfrak{X}$. It has shown that it can come from a finite group. These two association schemes have the same parameters but not isomorphic.

Example 1.9: Cyclotomic association scheme[BM90]

Assume that q is a prime number and $(q-1) = rd$. Let $GF(q)$ be a finite field with q elements. Then $GF(q)^* := GF(q)\backslash\{0\}$ is a cyclic group of order $q-1$. Let w be a generator of $GF(q)^*$. Denote the subgroup of order r of $GF(q)^*$ by $H_r =< w^d >$ and the cosets by $C_0 = \{0\}$, $C_i = w^{i-1}H_r$ for $1 \leqslant i \leqslant d$. Define $X = GF(q)$ and $R_i = \{(x,y) \in X \times X : x - y \in C_i\}$ for $0 \leqslant i \leqslant d$. Then $(GF(q), \{R_i\}_{0 \leqslant i \leqslant d})$ is a commutative association scheme. It is called the cyclotomic association scheme.

Proof. Let G be a set of maps from X to X defined by $G := \{\sigma : \sigma(x) = ax + b, a \in H_r, b \in GF(q)\}$. Then G form a group under composition and acts transitively on $GF(q)$ as $x^\sigma = \sigma(x)$. Each R_i becomes an orbit of G acting on $GF(q) \times GF(q)$. Since for every$(x,y) \in R_i$ and every $\sigma \in G$, $y^\sigma - x^\sigma = a(y-x) \in C_i$ while $y - x \in C_i$ and $a \in H_r$, so $(x^\sigma, y^\sigma) \in R_i$. By Example 1.3, $\mathfrak{X} = (GF(q), \{R_i\}_{0 \leqslant i \leqslant d})$ becomes an association scheme.

For commutativity, we show it as follows. For $(x,y) \in R_k$, i.e., $x - y \in C_k$, note that $|Y| = |-Y| = |Y + b|$ for every set Y, we have

$$p_{ij}^k = |\{z : x - z \in C_i, z - y \in C_j\}| = |(-C_i + x) \cap (C_j + y)|,$$

then

$$\begin{aligned} p_{ji}^k =& |(-C_j + x) \cap (C_i + y)| \\ =& |(C_j - x) \cap (-C_i - y)| \\ =& |(C_j + y) \cap (-C_i + x)| \\ =& |(-C_i + x) \cap (C_j + y)| = p_{ij}^k. \end{aligned}$$

□

Remark 1.10: Classification of association scheme of small sizes.

There are several results on classification of association scheme of small sizes.

- Izumi Miyamoto and Akihide Hanaki classify all the association schemes with $|X| \leqslant 30$: http://math.shinshu-u.ac.jp/ hanaki/as/.
- Other reference see [KTR13]. In particular, the association schemes with $|X| = 35, 36, 37, 39, 40$ are still open problems.

2 Bose-Mesner Algebra

Let X be a finite set and $R_i \subset X \times X$ be binary relations for $i \in \{0, 1, \ldots, d\}$. The matrix A_i is in $\mathbb{R}^{X \times X}$, whose rows and columns are parameterized by X. Define the entry of A_i by

$$(A_i)_{x,y} = A_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

We call A_i the **adjacency matrices** for R_i. The following conditions are equivalent to the conditions in Definition 1.1 :

(1′) $A_0 = I$, where I is the identity matrix;

(2′) $A_0 + A_1 + \cdots + A_d = J$ where J is the all one matrix;

(3′) ${}^tA_j = A_{j'}$ for some $j' = i \in \{0, 1, \ldots, d\}$;

(4′) for all $i, j \in \{0, 1, \ldots, d\}$ there exist non-negative integers p_{ij}^k where $k \in \{0, 1, \ldots, d\}$ such that $A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$;

(5′) $A_iA_j = A_jA_i$ for every $i, j \in \{0, 1, \ldots, d\}$;

(6′) ${}^tA_i = A_i$ for each $i \in \{0, 1, \ldots, d\}$.

Remark 2.1

Suppose that $A_0, \ldots, A_d$, all the $\{0, 1\}-$matrices in $M_{|X|}(\mathbb{C})$, satisfy (1′) to (4′). Then by defining

$$R_i = \{(x, y) \in X \times X : A_i(x, y) = 1\}.$$

We get an association scheme $(X, \{R_i\}_{0 \leqslant i \leqslant d})$.

Definition 2.2: Bose-Mesner Algebra

Let $A_0, A_1, \dots, A_d$ be the adjacency matrices of an association scheme $(X, \{R_i\}_{0 \leqslant i \leqslant d})$. Define the **Bose-Mesner Algebra** $\mathfrak{A}$ by

$$\mathfrak{A} = \{\sum_{i=0}^{d} a_i A_i : a_i \in \mathbb{C}\} \subset \mathrm{M}_{|\mathrm{X}|}(\mathbb{C}),$$

where $\mathrm{M}_{|\mathrm{X}|}(\mathbb{C})$ is the $|X|$ by $|X|$ matrix algebra over field $\mathbb{C}$. Since $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, $\mathfrak{A}$ is closed under ordinary matrix multiplication. The **Hadamard multiplication** between two matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant |X|}$ and $B = (b_{ij})_{1 \leqslant i,j \leqslant |X|}$ is defined by

$$A \circ B = (a_{ij} b_{ij})_{1 \leqslant i,j \leqslant |X|}.$$

Note that $A_i \circ A_j = \delta_{ij} A_i$, then $\mathfrak{A}$ is closed under Hadamard multiplication.

Lemma 2.3

Let $\mathfrak{M}$ be a linear subspace of $\mathrm{M}_{|\mathrm{X}|}(\mathbb{C})$. Suppose that $\mathfrak{M}$ is closed under Hadamard product. Then $\mathfrak{M}$ has a basis $A_0, \dots, A_d$ with $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leqslant i, j \leqslant d$.

Proof. Let $M = (m_{ij})_{0 \leqslant i,j \leqslant d} \in \mathfrak{M}$. List the non-zero numbers which appear in M as $\beta_1, \dots, \beta_r$, i.e., $\{\beta_1, \dots, \beta_r\} = \{m_{ij} : m_{ij} \neq 0, 0 \leq i, j \leq d\}$ with $\beta_1, \dots, \beta_r$ distinct. For each $i \in \{1, \dots, r\}$, write

$$M^{(i)}(x, y) := \begin{cases} 1 & \text{if } M(x, y) = \beta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Define $M^{\circ j} = \overbrace{M \circ \cdots \circ M}^{j}$. Then $M^{\circ j} = \sum_{i=1}^{r} \beta_i^j M^{(i)}$. Since

$$\begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_r \\ \beta_1^2 & \beta_2^2 & \cdots & \beta_r^2 \\ \vdots & \vdots & & \vdots \\ \beta_1^r & \beta_2^r & \cdots & \beta_r^r \end{vmatrix} \neq 0,$$

each $M^{(i)}$ is a linear combination of $M, M^{\circ 2}, \dots, M^{\circ r}$ for $i \in \{1, \dots, r\}$. In particular, $M^{(i)} \in \mathfrak{M}$.

Let $B_0, \dots, B_d$ be a basis of $\mathfrak{M}$. For each B_j, construct $B_j^{(1)}, \dots, B_j^{(r_j)}$. Then $B_j^{(i)} \in \mathfrak{M}$ and B_j is written as a linear combination of $B_j^{(1)}, \dots, B_j^{(r_j)}$. So $\{B_j^{(i)} : 1 \leqslant i \leqslant r_j, 0 \leqslant j \leqslant d\}$ span $\mathfrak{M}$.

Therefore, we can choose a basis $A_0, \dots, A_d$ of $\mathfrak{M}$ such that A_i is a zero-one matrix for each $i \in \{0, 1, \dots, d\}$. Suppose $A_i \circ A_j \neq 0$ for $i \neq j$. Then define $C_1 = A_i \circ A_j$ and $C_2 = A_i - C_1$. If necessary, change i and j, and assume that $C_1 \neq 0, C_2 \neq 0$. Since $C_1, C_2 \in \mathfrak{M}$, $\{A_0, A_1, \dots, A_{i-1}, C_1, C_2, A_{i+1}, \dots, A_d\}$ span $\mathfrak{M}$. Then we can choose a basis removing one element from this set. By this way, we can reduce the total number of 1's appearing in the basis elements. Repeating this reduction, we reach the desired basis. $\square$

We use I to denote the **identity matrix** and J to denote the **all one matrix**.

Lemma 2.4

Let $\mathfrak{M}$ be a subspace of $\mathrm{M}_{|\mathrm{X}|}(\mathbb{C})$ and let $\mathfrak{M}$ be closed under Hadamard product and ordinary matrix product. Moreover, suppose the following conditions hold:

(I) If $M \in \mathfrak{M}$, then ${}^tM \in \mathfrak{M}$;

(II) For each $M \in \mathfrak{M}$, there exists a constant $\alpha(M, I)$, such that $M \circ I = \alpha(M, I)I$;

(III) $I, J \in \mathfrak{M}$.

Then there exists a set of $\{0, 1\}$-matrices $A_0, \ldots, A_d$ form a basis of $\mathfrak{M}$ satisfying the conditions (1′) to (4′).

Proof. By Lemma 2.3, $\mathfrak{M}$ has a basis $A_0, \ldots, A_d$ of $\{0, 1\}$-matrices with $A_i \circ A_j = \delta_{ij} A_i$. Check the (1′) to (4′):

(1′) Since $I \in \mathfrak{M}$, $I = \sum_{i=0}^{d} a_i A_i$. Since $I \circ I = I$, $I \circ I = \sum_{i=0}^{d} a_i^2 A_i = I$, i.e., $a_i^2 = a_i$, then we have either $a_i = 0$ or $a_i = 1$. By the condition (II), $\alpha(A_j, I)I = A_j \circ I = A_j$, then there exist j such that $A_j = I$. By changing the order, we may assume that $A_0 = I$.

(2′) Let $J = \sum_{i=0}^{d} b_i A_i$. Then $A_j = A_j \circ J = \sum_{i=0}^{d} b_i (A_j \circ A_i) = \sum_{i=0}^{d} b_i (\delta_{ij} A_i) = b_i A_j$, for all $j \in \{1, \ldots, d\}$. Hence $b_0 = b_1 = \cdots = b_d = 1$. It satisfies that $J = A_0 + \cdots + A_d$.

(3′) Since ${}^tJ = J$ and $A_0 + \cdots + A_d = {}^tA_0 + \cdots + {}^tA_d$, then ${}^tA_i$ is a sum of $A_0, \ldots, A_d$. Note that ${}^tA_i\, {}^tA_j = {}^t(A_i A_j) = \delta_{ij}\, {}^tA_i$. Then there exists an one-to-one correspondence such that ${}^tA_i = A_j$ for $j \in \{0, 1, \ldots, d\}$.

(4′) By the assumption, $\mathfrak{M}$ is closed under ordinary multiplication, that is, $A_i A_j \in \mathfrak{M}$, $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$.

□

For further discuss, we need to introduce some notations. Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative association scheme. For each $x \in X$, we define $\Gamma_i(x) := \{z \in X : (x, z) \in R_i\}$ and $\Gamma_{i'}(x) := \{z \in X : (z, x) \in R_i\}$. For all $(x, y) \in R_k$, note that $|\Gamma_i(x) \cap \Gamma_{j'}(y)| = p_{ij}^k$. We define the **valency** of the relation R_i to be the non-negative integer $k_i := p_{ii'}^0$. By the definition, for every $x \in X$ we have $(x, z) \in R_i \Leftrightarrow (z, x) \in R_{i'}$. Hence $k_i = |\Gamma_i(x)|$.

Proposition 2.5

Let $\mathfrak{X}$ be a commutative association scheme, we have

(1) $k_0 = 1$;

(2) $k_i = k_{i'}$;

(3) $|X| = k_0 + \cdots + k_d$.

Proof. (1) $k_0 = |\Gamma_0(x)| = |\{z \in X : (x, z) \in R_0\}| = 1$.

(2) By the commutativity, $k_i = p_{ii'}^0 = p_{i'i}^0 = k_{i'}$.

(3) Take $x \in X$. Since $X \times X = R_0 \cup \cdots \cup R_d$, then

$$|X| = \left| \bigcup_{i=0}^{d} \Gamma_i(x) \right| = k_0 + \cdots + k_d.$$

□

Proposition 2.6

Given a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$, we have

(1) $p_{i0}^k = \delta_{ik}$;

(2) $p_{0j}^k = \delta_{jk}$;

(3) $p_{ij}^0 = k_i\delta_{ij'}$;

(4) $p_{ij}^k = p_{i'j'}^{k'}$;

(5) $\sum_{j=0}^d p_{ij}^k = k_i$;

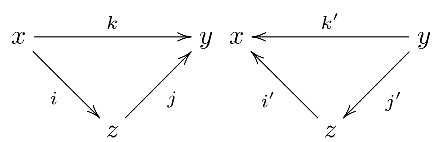
(6) $k_\ell p_{ij}^\ell = k_j p_{i'\ell}^j = k_i p_{\ell j'}^i$;

(7) $\sum_{\alpha=0}^d p_{ij}^\alpha p_{k\alpha}^\ell = \sum_{\beta=0}^d p_{ki}^\beta p_{\beta j}^\ell$.

Proof. (1) For all $(x, y) \in R_k$, we have $p_{i0}^k = |\{z \in X : (x, z) \in R_i, (z, y) \in R_0\}| = \delta_{ik}$.

(2) For all $(x, y) \in R_k$, we have $p_{0j}^k = |\{z \in X : (x, z) \in R_0, (z, y) \in R_j\}| = \delta_{jk}$.

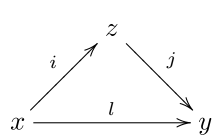
(3) Take $(x, x) \in R_0$, we have $p_{ij}^0 = |\{z \in X : (x, z) \in R_i, (z, x) \in R_j\}|$ and $\Gamma_i(x) = \{z \in X : (x, z) \in R_i\}$. Hence $p_{ij}^0 = |\Gamma_i(x)|\delta_{ij'} = k_i\delta_{ij'}$.



(4) By definition, we have
$$p_{ij}^k = |\{z : (x, z) \in R_i, (z, y) \in R_j\}|$$
and
$$p_{j'i'}^{k'} = |\{z' : (z', x') \in R_j, (y', z') \in R_i\}|,$$
where $(x, y) \in R_k$ and $(y', x') \in R_k$. Hence $p_{ij}^k = p_{j'i'}^{k'}$. By the commutativity, $p_{ij}^k = p_{j'i'}^{k'} = p_{i'j'}^{k'}$.

(5) Take $(x, y) \in R_k$. Note that $\Gamma_i(x) = \bigcup_{j=0}^d \big(\Gamma_i(x) \cap \Gamma_j(y)\big)$. And then $k_i = \sum_{j=0}^d p_{ij}^k$.

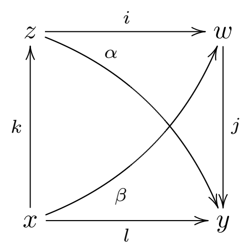
(6) Count the number of $(x, y, z) \in X^3$ which satisfies the relations in the following figure:



We have
$$\begin{aligned} &\big\{(x, y, z) : (x, y) \in R_\ell, (x, z) \in R_i, (z, y) \in R_j\big\} \\ =& \bigcup_{x\in X} \bigcup_{y\in\Gamma_\ell(x)} \big\{(x, y, z) : (x, z) \in R_i, (z, y) \in R_j\big\} \\ =& \bigcup_{z\in X} \bigcup_{y\in\Gamma_j(z)} \big\{(x, y, z) : (z, x) \in R_{i'}, (x, y) \in R_\ell\big\} \\ =& \bigcup_{x\in X} \bigcup_{z\in\Gamma_i(x)} \big\{(x, y, z) : (x, y) \in R_\ell, (y, z) \in R_{j'}\big\} \end{aligned}$$

Then $|X|k_l p_{ij}^l = |X|k_j p_{i'l}^j = |X|k_i p_{lj'}^i$, it follows that $k_l p_{ij}^l = k_j p_{i'l}^j = k_i p_{lj'}^i$.

(7) Fix $(x, y) \in R_k$, count the number of (z, w) which satisfies the relations in the following figure:



We have

$$\begin{aligned}
&\left\{(z,w) : (z,w) \in R_i, (x,z) \in R_k, (w,y) \in R_j\right\} \\
=& \bigcup_{\alpha=0}^{d} \bigcup_{z \in \Gamma_k(x) \cap \Gamma_{\alpha'}(y)} \left\{(z,w) : (z,w) \in R_i, (w,y) \in R_j\right\} \\
=& \bigcup_{\beta=0}^{d} \bigcup_{w \in \Gamma_\beta(x) \cap \Gamma_{j'}(y)} \left\{(z,w) : (x,z) \in R_k, (z,w) \in R_i\right\}
\end{aligned}$$

It follows that $\sum_{\alpha=0}^{d} p_{ij}^{\alpha} p_{k\alpha}^{l} = \sum_{\beta=0}^{d} p_{ki}^{\beta} p_{\beta j}^{l}$.

□

Lemma 2.7

Let $\mathfrak{M} \subset \mathrm{M}_{|\mathrm{X}|}(\mathbb{C})$ be a vector space. Suppose $\mathfrak{M}$ is closed under ordinary matrix multiplication and $\mathfrak{M}$ is commutative. Furthermore, we suppose that

(1) $M \in \mathfrak{M} \implies {}^tM, \overline{M} \in \mathfrak{M}$;

(2) For any $M \in \mathfrak{M}$, there exists a fixed real number $\alpha(M, J)$ such that $JM = \alpha(M, J)J$;

(3) $I, J \in \mathfrak{M}$.

Then there exists a basis $E_0 (= \frac{1}{|X|} J), E_1, \ldots, E_d$ satisfying the following 3 conditions:

($1''$) $|X|\, E_0 = J$;

($2''$) $E_0 + E_1 + \cdots + E_d = I, E_i E_j = \delta_{ij} E_i$;

($3''$) For each $i \in \{0, 1, \ldots, d\}$, ${}^tE_i = E_{\hat{i}}$ for some $\hat{i} \in \{0, 1, \ldots, d\}$.

Proof. For any $M \in \mathfrak{M}$, we have $M\,\overline{{}^tM} = \overline{{}^tM}\,M$. So M is a normal matrix, thus $\mathfrak{M}$ has a basis consisting of normal matrices, which are mutually commutative. From linear algebra, we know that they can be diagonalized by unitary matrix simultaneously. That is to say, there exists a unitary matrix U such that $\overline{{}^tU} M U$ are all diagonal matrices. Among diagonal matrices, ordinary matrix multiplication and Hadamard multiplication are the same. So $\overline{{}^tU}\,\mathfrak{M}\,U$ is closed under Hadamard multiplication. By Lemma 2.3, there is a basis $\Lambda_0, \Lambda_1, \ldots, \Lambda_d$ with $\Lambda_i \circ \Lambda_j = \delta_{ij}\Lambda_i$. We define E_i to be $U \Lambda_i \overline{{}^tU}$, then $E_0, E_1, \ldots, E_d$ is a basis of $\mathfrak{M}$ and $E_i E_j = \delta_{ij} E_i$. Let $J = |X| \sum_{j=0}^{d} a_j E_j$. Since $J^2 = |X|\, J = |X|^2 \sum_{j=0}^{d} a_j E_j$ and also $J^2 = \sum_{j=0}^{d} a_j^2 E_j$. Compare the coefficients, we have $a_j = a_j^2$, i.e., $a_j = 0$ or 1. Suppose $a_j \neq 0$ (there is at least one such j), then

$$|X|\, E_j = J E_j = \alpha(J, E_j) J = \alpha(J, E_j)\, |X| \sum_{j=0}^{d} a_j E_j.$$

Hence $J = |X|\, E_j$ for some j. By changing the ordering of $E_0, E_1, \ldots, E_d$, we may assume $j = 0$, i.e., $E_0 = \frac{1}{|X|}J$. Then (1″) followss Suppose $I = \sum_{j=0}^{d} b_j E_j$, then $I = I^2 = \sum_{j=0}^{d} b_j^2 E_j$. Thus $b_j = 0$ or 1. In fact $b_0 = b_1 = \cdots = b_d = 1$. Otherwise, say $b_j = 0$, then we have $E_j = IE_j = 0E_j = 0$, contradiction. Note that

$$E_0 + E_1 + \cdots + E_d = I = {}^tI = {}^tE_0 + {}^tE_1 + \cdots + {}^tE_d$$

and $({}^tE_i)({}^tE_j) = \delta_{ij}({}^tE_i)$. Since each ${}^tE_i$ is a linear combination of $E_0, E_1, \ldots, E_d$, so ${}^tE_i = E_{\hat{i}}$ for some $\hat{i} \in \{0, 1, \ldots, d\}$. Hence we obtain (2″) and (3″) □

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative association scheme, and $A_0, A_1, \ldots, A_d$ be the adjacency matrices of the relations respectively. The Bose-Mesner algebra $\mathfrak{A}$ is nothing but the algebra spanned by these matrices.

Lemma 2.7 implies that there exists another basis $E_0, E_1, \ldots, E_d$ of $\mathfrak{A}$ such that $I = E_0 + E_1 + \cdots + E_d$ and $E_iE_j = \delta_{ij}E_i$. These E_i's are called primitive idempotents. Since $\mathfrak{A}$ is closed under Hadamard multiplication, we have the followings:

(4″) $E_i \circ E_j = \frac{1}{|X|}\sum_{j=0}^{d} q_{ij}^k E_k$ where $q_{ij}^k \in \mathbb{C}$. These q_{ij}^k's are called **Krein parameters**;

(5″) $E_i \circ E_j = E_j \circ E_i$.

Remark 2.8

For unique $\hat{i} \in \{0, \ldots, d\}$, $\overline{E_i} = {}^tE_i = E_{\hat{i}}$.

Proof. Note that ${}^tA_i \in \mathfrak{A}$ and $\overline{A_i} = A_i \in \mathfrak{A}$, so $\overline{{}^tE_i} \in \mathfrak{A}$ and $\left(\overline{{}^tE_i}\right)\left(\overline{{}^tE_j}\right) = \delta_{ij}\,\overline{{}^tE_i}$. Let $V = \mathbb{C}\,X$ considered as column vectors. For any $0 \neq u \in V_i := E_iV$, we have

$$0 \neq \langle u, u\rangle = \langle E_iu, E_iu\rangle = {}^t(E_iu)\overline{(E_iu)} = {}^tu\,{}^tE_i\overline{E_iu} = \left\langle u, \overline{{}^tE_i}E_iu\right\rangle$$

where $\langle , \rangle$ is the usual inner product. So $\overline{{}^tE_i}E_i \neq 0$, thus $\overline{E_i} = {}^tE_i$. □

3 The Character Table of Commutative Association Scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative association scheme and $V = \mathbb{C}\,X$ identified with the space of column vectors. We have $I = E_0 + E_1 + \cdots + E_d$ and $E_iE_j = \delta_{ij}E_i$. Then $V = V_0 \perp V_1 \perp \cdots \perp V_d$ where $V_i = E_iV$. In particular, $E_0 = \frac{1}{|X|}J$. For any $M \in \mathfrak{M}$, we have $ME_i = (\text{constant})E_i$, so each subspace $V_i = E_iV$ is a common eigenspace of $\mathfrak{A}$. Let $P_i(j) \in \mathbb{C}$ be the scalar of $A_i \in \mathfrak{A}$ on V_j, i.e.,

$$A_i = \sum_{j=0}^{d} P_i(j)E_j, \tag{1}$$

for each $i \in \{0, 1, \ldots, d\}$. The matrix

$$P = (P_j(i))_{0 \leqslant i,j \leqslant d} = (P_{ij})$$

is called the **character table** or **the first eigenmatrix** of $\mathfrak{X}$. Similarly, we have

$$E_i = \frac{1}{|X|}\sum_{j=0}^{d} Q_i(j)A_j. \tag{2}$$

The matrix

$$Q = (Q_j(i))_{0 \leqslant i,j \leqslant d} = (Q_{ij})$$

is called **the second eigenmatrix** of $\mathfrak{X}$. They satisfy the obvious identity

$$PQ = QP = |X|\, I. \tag{3}$$

Proposition 3.1

Let $m_i = \dim V_i = \operatorname{rank} E_i$, then we have the following identities:

(1) $\operatorname{tr}(A_i) = \begin{cases} |X| & i = 0; \\ 0 & \text{otherwise}; \end{cases}$

(2) $JA_i = A_iJ = k_iJ$;

(3) $\operatorname{tr}(E_i) = m_i$.

Proof. (1) Because $A_0 = I$ and $A_i(x, x) = 0$ for any $i \neq 0$.

(2) $(JA_i)(x, y) = \sum_{z \in X} J(x, z)A_i(z, y) = \sum_{z \in X} A_i(z, y) = |\{z : (z, y) \in R_i\}| = k_i$.

(3) Since $E_i^2 = E_i$, the eigenvalue of E_i is 0 or 1. Diagonalize E_i and we have $\operatorname{tr}(E_i) = \operatorname{rank} E_i = m_i$. □

Proposition 3.2

For every $i \in \{0, 1, \dots, d\}$, we have the followings:

(1) $P_0(i) = 1$;

(2) $P_i(0) = k_i$;

(3) $Q_0(i) = 1$;

(4) $Q_i(0) = m_i$.

Proof. (1) $A_0 = I = E_0 + E_1 + \cdots + E_d$.

(2) By Proposition 3.1, we have $E_0A_i = \frac{1}{|X|}JA_i = \frac{1}{|X|}k_iJ = k_iE_0$. By Eq. (1), we have $E_0A_i = E_0 \sum_{j=0}^{d} P_i(j)E_i = P_i(0)E_0$. Hence $P_i(0) = k_i$ for each i.

(3) Compare $E_0 = \frac{1}{|X|}J = \frac{1}{|X|}(A_0 + A_1 + \cdots + A_d)$ with $E_0 = \sum_{j=0}^{d} Q_0(j)A_j$.

(4) By Proposition 3.1, we have $m_i = \operatorname{tr}(E_i)$. By Eq. (1), we have $\frac{1}{|X|} \sum_{j=0}^{d} Q_i(j) \operatorname{tr}(A_j) = Q_i(0)$.

□

Theorem 3.3

Let $\mathfrak{X}$ be a commutative association scheme, we have

(1) $P_{i'}(j) = \overline{P_i(j)}$;

(2) $Q_{\hat{i}}(j) = \overline{Q_i(j)}$;

(3) $\dfrac{Q_j(i)}{m_j} = \dfrac{\overline{P_i(j)}}{k_i}$;

(4) $\sum_{\gamma=0}^{d} \dfrac{1}{k_\gamma} P_\gamma(i)\overline{P_\gamma(j)} = \delta_{ij}\dfrac{|X|}{m_i}$ (**the first orthogonal relation**);

(5) $\sum_{\gamma=0}^{d} m_\gamma P_i(\gamma)\overline{P_j(\gamma)} = \delta_{ij}\,|X|\,k_i$ (**the second orthogonal relation**);

(6) $P_i(\ell)P_j(\ell) = \sum_{k=0}^{d} p_{ij}^k P_k(\ell)$;

(7) $Q_i(\ell)Q_j(\ell) = \sum_{k=0}^{d} q_{ij}^k Q_k(\ell)$;

(8) $P_i(j)Q_j(\ell) = \sum_{k=0}^{d} p_{ik}^{\ell} Q_j(k)$.

Proof. (1) $A_{i'} = {}^tA_i = \overline{{}^tA_i} = \sum_{j=0}^{d} \overline{P_i(j)}\,\overline{{}^tE_j} = \sum_{j=0}^{d} \overline{P_i(j)}E_j$. So $P_{i'}(j) = \overline{P_i(j)}$.

(2) $\frac{1}{|X|}\sum_{j=0}^{d} Q_{\hat{i}}(j)A_j = E_{\hat{i}} = {}^tE_i = \overline{E_i} = \frac{1}{|X|}\sum_{j=0}^{d} \overline{Q_i(j)}A_j$. So $Q_{\hat{i}}(j) = \overline{Q_i(j)}$.

(3) $A_i \circ E_j = \frac{1}{|X|}Q_j(i)A_i$. So

$$\begin{aligned}\sum_{x\in X}\sum_{y\in X}(A_i\circ E_j)(x,y) &= \sum_{x\in X}\sum_{y\in X}(A_i(x,y)E_j(x,y))\\ &= \sum_{x\in X}(E_j\,{}^tA_i)(x,x)\\ &= \operatorname{tr}(E_jA_{i'})\\ &= \operatorname{tr}(P_{i'}(j)E_j)\\ &= P_{i'}(j)m_j = m_j\overline{P_i(j)}\end{aligned}$$

while $\displaystyle\sum_{x\in X}\sum_{y\in X}\left(\frac{1}{|X|}Q_j(i)A_i\right) = k_iQ_j(i)$.

(4) Calculate the (i,j)-entry of $PQ = |X|\,I$, then use (3):

$$|X|\,\delta_{ij} = \sum_{\gamma=0}^{d} P_\gamma(i)Q_j(\gamma) = \sum_{\gamma=0}^{d} P_\gamma(i)\frac{m_j}{k_\gamma}\overline{P_j(\gamma)}.$$

(5) Calculate the (i,j)-entry of $QP = |X|\,I$, then use (3):

$$|X|\,\delta_{ij} = \sum_{\gamma=0}^{d} Q_\gamma(i)P_j(\gamma) = \sum_{\gamma=0}^{d} \frac{m_\gamma}{k_i}\overline{P_i(\gamma)}P_j(\gamma).$$

Apply Conjugation on both sides.

(6) Compare

$$A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k = \sum_{k=0}^{d} p_{ij}^k \sum_{\ell=0}^{d} P_k(\ell)E_\ell = \sum_{\ell=0}^{d}\sum_{k=0}^{d} p_{ij}^k P_k(\ell)E_\ell$$

with

$$A_iA_j = \left(\sum_{\ell=0}^{d} P_i(\ell)E_\ell\right)\left(\sum_{k=0}^{d} P_i(k)E_k\right) = \sum_{\ell=0}^{d} P_i(\ell)P_j(\ell)E_\ell.$$

(7) Compare

$$E_i \circ E_j = \frac{1}{|X|}\sum_{k=0}^{d} q_{ij}^k E_k = \frac{1}{|X|}\sum_{k=0}^{d} q_{ij}^k \frac{1}{|X|}\sum_{\ell=0}^{d} Q_k(\ell)A_\ell = \frac{1}{|X|^2}\sum_{\ell=0}^{d}\sum_{k=0}^{d} q_{ij}^k Q_k(\ell)A_\ell$$

with

$$E_i \circ E_j = \left(\frac{1}{|X|}\sum_{\ell=0}^{d} Q_i(\ell)A_\ell\right) \circ \left(\frac{1}{|X|}\sum_{k=0}^{d} Q_j(k)A_k\right) = \frac{1}{|X|^2}\sum_{\ell=0}^{d} Q_i(\ell)Q_j(\ell)A_\ell.$$

(8) Compare

$$A_iE_j = A_i\frac{1}{|X|}\sum_{k=0}^{d} Q_k(\ell)A_k = \frac{1}{|X|}\sum_{k=0}^{d} Q_j(k)\sum_{\ell=0}^{d} p_{ik}^\ell A_\ell = \frac{1}{|X|}\sum_{\ell=0}^{d}\sum_{k=0}^{d} p_{ik}^\ell Q_j(k)A_\ell$$

with

$$A_iE_j = P_i(j)E_j = P_i(j)\frac{1}{|X|}\sum_{k=0}^{d} Q_j(\ell)A_\ell = \frac{1}{|X|}\sum_{\ell=0}^{d} P_i(j)Q_j(\ell)A_\ell.$$

□

Theorem 3.4

Let $\mathfrak{X}$ be a commutative association scheme, then we have:

(1) $q_{ij}^\ell = \dfrac{m_im_j}{|X|}\displaystyle\sum_{\gamma=0}^{d}\frac{1}{k_\gamma^2}P_\gamma(i)P_\gamma(j)\overline{P_\gamma(\ell)}$;

(2) $p_{ij}^\ell = \dfrac{k_ik_j}{|X|}\displaystyle\sum_{\gamma=0}^{d}\frac{1}{m_\gamma^2}Q_\gamma(i)Q_\gamma(j)\overline{Q_\gamma(\ell)}$.

Proof. (1) Consider the trace of $\frac{1}{|X|}q_{ij}^{\ell}E_\ell = (E_i \circ E_j)E_\ell$.

$$\begin{aligned}
\text{LHS} &= \frac{1}{|X|}q_{ij}^{\ell}m_\ell \\
\text{RHS} &= \operatorname{tr}((E_i \circ E_j)E_\ell) = \sum_{x\in X}((E_i \circ E_j)E_\ell)(x,x) \\
&= \sum_{x\in X}\sum_{y\in X} E_i(x,y)E_j(x,y)E_\ell(y,x) \\
&= \sum_{x\in X}\sum_{y\in X} (E_i \circ E_j \circ {}^tE_\ell)(x,y) \\
&= \sum_{x\in X}\sum_{y\in X}\left(\frac{1}{|X|^2}\sum_{\gamma=0}^{d} Q_i(\gamma)Q_j(\gamma)Q_{\hat{\ell}}(\gamma)A_\gamma(x,y)\right) \\
&= \frac{1}{|X|^2}\sum_{\gamma=0}^{d} Q_i(\gamma)Q_j(\gamma)\overline{Q_l(\gamma)}k_\gamma \\
&= \frac{1}{|X|^2}\sum_{\gamma=0}^{d} \frac{m_i}{k_\gamma}\overline{P_\gamma(i)}\frac{m_j}{k_\gamma}\overline{P_\gamma(j)}\overline{\frac{m_\ell}{k_\gamma}\overline{P_\gamma(\ell)}}k_\gamma \\
&= \frac{m_i m_j m_\ell}{|X|^2}\sum_{\gamma=0}^{d}\frac{1}{k_\gamma^2}\overline{P_\gamma(i)P_\gamma(j)\overline{P_\gamma(\ell)}}.
\end{aligned}$$

Compare the LHS and RHS, we only need to show that $q_{ij}^{\ell} \in \mathbb{R}$ (in fact $q_{ij}^{\ell} \geqslant 0$).

$$\begin{aligned}
&\sum_{\gamma=0}^{d}\frac{1}{k_\gamma^2}P_\gamma(i)P_\gamma(j)\overline{P_\gamma(\ell)} \\
=&\sum_{\gamma=0}^{d}\frac{1}{k_{\gamma'}^2}P_{\gamma'}(i)P_{\gamma'}(j)\overline{P_{\gamma'}(\ell)} \\
=&\sum_{\gamma=0}^{d}\frac{1}{k_\gamma^2}\overline{P_\gamma(i)P_\gamma(j)\overline{P_\gamma(\ell)}}
\end{aligned}$$

So $q_{ij}^{\ell} \in \mathbb{R}$.

(2) Calculate the sum of the entries of $p_{ij}^{\ell} A_l = (A_i A_j) \circ A_l$.

$$\begin{aligned}
\text{LHS} &= |X|\, p_{ij}^{\ell} k_\ell \\
\text{RHS} &= \sum_{x\in X}\sum_{y\in X} \left((A_iA_j)\circ A_\ell\right)(x,y) \\
&= \sum_{x\in X}\sum_{y\in X} (A_iA_j)(x,y)\, {}^tA_\ell(y,x) \\
&= \operatorname{tr}(A_iA_jA_{\ell'}) \\
&= \operatorname{tr}\left(\sum_{\gamma=0}^{d} P_i(\gamma)P_j(\gamma)P_{\ell'}(\gamma)E_\gamma\right) \\
&= \sum_{\gamma=0}^{d} P_i(\gamma)P_j(\gamma)P_{\ell'}(\gamma)m_\gamma \\
&= \frac{k_ik_jk_\ell}{m_\gamma^2}\overline{Q_\gamma(i)Q_\gamma(j)}Q_\gamma(\ell) \\
&= \frac{1}{|X|}m_i\sum_{k=0}^{d} E_k.
\end{aligned}$$

Since LHS is real, we apply conjugation on both sides. □

Proposition 3.5

Krein parameters q_{ij}^k's satisfy the following relations:

(1) $q_{0j}^k = \delta_{jk}$;

(2) $q_{i0}^k = \delta_{ik}$;

(3) $q_{ij}^0 = \delta_{i\hat{j}} m_i$;

(4) $q_{ij}^k = q_{\hat{i}\hat{j}}^{\hat{k}}$;

(5) $\sum_{j=0}^{k} q_{ij}^k = m_i$;

(6) $m_k q_{ij}^k = m_j q_{\hat{i}k}^j = m_i q_{k\hat{j}}^i$;

(7) $\sum_{\alpha=0}^{d} q_{ij}^\alpha q_{k\alpha}^\ell = \sum_{\beta=0}^{d} q_{ki}^\beta q_{\beta j}^\ell$.

Proof. (1) $\frac{1}{|X|}E_j = E_0 \circ E_j = \frac{1}{|X|}\sum_{k=0}^{d} q_{0j}^k E_k$.

(2) $\frac{1}{|X|}E_i = E_i \circ E_0 = \frac{1}{|X|}\sum_{k=0}^{d} q_{i0}^k E_k$.

(3)

$$
\begin{aligned}
q_{ij}^k &= \operatorname{tr}\left(q_{ij}^k E_0\right) = \operatorname{tr}\left(\sum_{k=0}^{d} q_{ij}^k E_k E_0\right) = |X| \operatorname{tr}\left((E_i \circ E_j) E_0\right) \\
&= |X| \sum_{x\in X}\sum_{y\in X} (E_i \circ E_j)(x,y) E_0(y,x) \\
&= \sum_{x\in X}\sum_{y\in X} (E_i \circ E_j)(x,y) \\
&= \operatorname{tr}(E_i\, {}^t E_j) = \operatorname{tr}(E_i E_{\hat{j}}) = \operatorname{tr}(\delta_{i\hat{j}} E_i) \\
&= \delta_{i\hat{j}} m_i.
\end{aligned}
$$

(4) Take the transpose of $E_i \circ E_j = \frac{1}{|X|}\sum_{j=0}^{d} q_{ij}^k E_k$.

(5) Take the sum of $E_i \circ E_j = \frac{1}{|X|}\sum_{j=0}^{d} q_{ij}^k E_k$ over j, then

$$
\begin{aligned}
\text{LHS} &= E_i \circ A_0 \\
&= \left(\frac{1}{|X|}\sum_{l=0}^{d} Q_i(l) A_l\right) \circ A_0 \\
&= \frac{1}{|X|} Q_i(0) A_0 \\
&= \frac{1}{|X|} m_i \sum_{k=0}^{d} E_k \\
&= \frac{1}{|X|} \sum_{k=0}^{d} m_i E_k \\
\text{RHS} &= \frac{1}{|X|} \sum_{k=0}^{d}\sum_{j=0}^{d} q_{ij}^k E_k.
\end{aligned}
$$

(6)

$$
\begin{aligned}
m_j q_{ik}^j &= m_j \frac{m_i m_k}{|X|} \sum_{\gamma=0}^{d} \frac{1}{k_\gamma^2} P_\gamma(\hat{i}) P_\gamma(k) \overline{P_\gamma(j)} \\
&= m_j \frac{m_i m_k}{|X|} \sum_{\gamma=0}^{d} \frac{1}{k_\gamma^2} \overline{P_\gamma(i)} P_\gamma(k) \overline{P_\gamma(j)} \\
&= m_j \frac{m_i m_k}{|X|} \sum_{\gamma=0}^{d} \frac{1}{k_\gamma^2} P_{\gamma'}(i) P_\gamma(k) P_{\gamma'}(j) \\
&= m_k \frac{m_i m_j}{|X|} \sum_{\gamma'=0}^{d} \frac{1}{k_{\gamma'}^2} P_{\gamma'}(i) P_{\gamma'}(j) \overline{P_{\gamma'}(k)} \\
&= m_k q_{ij}^k.
\end{aligned}
$$

(7) Note that $E_k \circ (E_i \circ E_j) = (E_k \circ E_i) \circ E_j$, we have

$$
\text{LHS} = E_k \circ \left(\frac{1}{|X|}\sum_{\alpha=0}^{d} q_{ij}^\alpha E_\alpha\right) = \frac{1}{|X|^2}\sum_{\ell=0}^{d}\sum_{\alpha=0}^{d} q_{ij}^\alpha q_{k\alpha}^\ell E_\ell
$$

and

$$\text{RHS} = \left(\frac{1}{|X|} \sum_{\beta=0}^{d} q_{ki}^{\beta} E_\beta \right) \circ E_j = \frac{1}{|X|^2} \sum_{\ell=0}^{d} \sum_{\beta=0}^{d} q_{ki}^{\beta} q_{\beta j}^{\ell} E_\ell. \qquad \square$$

Theorem 3.6: Krein condition

$q_{ij}^k \geqslant 0$.

Proof. E_i and E_j are non-negative definite Hermitian matrices, so $E_i \otimes E_j$ is also non-negative definite. Note that $E_i \circ E_j$ is a principal minor of $E_i \otimes E_j$, thus it is also a non-negative definite Hermitian matrix. So the eigenvalue q_{ij}^k of $E_i \circ E_j$ is non-negative. $\square$

Remark 3.7

$\mathfrak{X}$ is a symmetric association scheme if and only if $P_j(i) \in \mathbb{R}$ for any $i, j \in \{0, 1, \ldots, d\}$.

Proof. "$\Rightarrow$"

Since $i = i'$, we have $\overline{P_i(j)} = P_{i'}(j) = P_i(j)$. Hence $P_j(i) \in \mathbb{R}$.

"$\Leftarrow$"

Suppose $P_j(i) \in \mathbb{R}$ for any $i, j \in \{0, 1, \ldots, d\}$ and suppose ${}^tR_s = R_{s'}$ where $s \neq s'$. Since $P_{s'}(j) = \overline{P_s(j)} = P_s(j)$, by the second orthogonal relation we have

$$0 \neq \sum_{\gamma=0}^{d} m_\gamma P_s(\gamma) P_{s'}(\gamma) = \delta_{ss'} |X| k_s = 0.$$

Contradiction. $\square$

Question 3.1

For any commutative association scheme, are all values of $P_j(i)$ in a cyclotomic number field? (Note that $P_j(i)$'s are algebraic integers).

Theorem 3.8: [Mun91]

$$\frac{\mathbb{Q}(\{P_j(i)\})}{\mathbb{Q}\left(\{q_{ij}^k\}\right)} \subseteq Z\left(\mathrm{Gal}\left(\frac{\mathbb{Q}(\{P_j(i)\})}{\mathbb{Q}\left(\{q_{ij}^k\}\right)}\right)\right).$$

How to find P and Q from p_{ij}^k's?

Definition 3.9: Intersection matrices

$B_i := \left(p_{ij}^k\right)_{0 \leqslant j,k \leqslant d}$. Equivalently $B_i(j,k) = p_{ij}^k$. These $(d+1) \times (d+1)$ matrices are called **intersection matrices**.

Theorem 3.10

Let $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle$ be the Bose-Mesner algebra. Then the map $A_i \mapsto B_i, i \in \{0, 1, \ldots, d\}$ gives an algebra isomorphism from $\mathfrak{A}$ to $\mathfrak{B} = \langle B_0, B_1, \ldots, B_d \rangle \subset M_{d+1}(\mathbb{C})$.

Proof. For each $M \in \mathfrak{A}$, let $L_M \in \mathrm{End}(\mathfrak{A})$ be the map defined by $L_M(N) = MN$, for all $N \in \mathfrak{A}$. Let $\rho(M)$ be the matrix representation of M with resepect to the basis $A_0, A_1, \ldots, A_d$. $\rho : \mathfrak{A} \to M_{d+1}(\mathbb{C})$

$$(MA_0, MA_1, \ldots, MA_d) = (A_0, A_1, \ldots, A_d)\rho(M).$$

This map is called the left regular representation of $\mathfrak{A}$. $\rho(M) = 0 \implies L_M(N) = 0$, for all $N \in \mathfrak{A}$. In particular $L_M(I) = MI = 0 \implies M = 0$. So ρ is injective. Since $A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$, we have

$$(MA_0, MA_1, \ldots, MA_d) = (A_0, A_1, \ldots, A_d)\, {}^tB_i,$$

i.e., $\rho(A_i) = {}^tB_i$. So $A_i \mapsto B_i = {}^t(\rho(A_i))$ gives an algebra isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. □

Remark 3.11

The minimum polynomial of A_i and the minimum polynomial of B_i are the same.

We also have Krein parameters q_{ij}^k which satisfy $(|X|\, E_i) \circ (|X|\, E_j) = \sum_{k=0}^{d} q_{ij}^k (|X|\, E_k)$.

Definition 3.12

$B_i^* := (q_{ij}^k)_{0 \leqslant j,k \leqslant d}$. Equivalently $B_i^*(j,k) = q_{ij}^k$. These matrices are called **dual intersection matrices**. They generate an algebra $\mathfrak{B}^* = \langle B_0^*, B_1^*, \ldots, B_d^* \rangle$.

Theorem 3.13

The map $|X|\, E_i \mapsto B_i^*$ gives an algebra isomorphism from $\mathfrak{A}^\circ$ to $\mathfrak{B}^*$ where $\mathfrak{A}^\circ$ is the algebra with the Hadamard multiplication.

Proof. For each $M \in \mathfrak{A} = \mathfrak{A}^\circ$, define $L_M^* \in \mathrm{End}(\mathfrak{A}^\circ)$ by $L_M^*(N) = M \circ N$ for $\forall N \in \mathfrak{A}$. Take a basis $|X|\, E_0, |X|\, E_1, \ldots, |X|\, E_d$, and represent L_M^* as a matrix representation using this basis:

$$(M \circ (|X|\, E_0), M \circ (|X|\, E_1), \ldots, M \circ (|X|\, E_d)) = (|X|\, E_0, |X|\, E_1, \ldots, |X|\, E_d)\rho^*(M)$$

Since $(|X|\, E_i) \circ (|X|\, E_j) = \sum_{k=0}^{d} q_{ij}^k (|X|\, E_k)$, we have $\rho^*(|X|\, E_i) = {}^tB_i^*$. Note that $\rho^*(M) = 0 \implies L_M^*(N) = 0, \forall N \in \mathfrak{A}$. In particular $L_M(J) = M \circ J = 0 \implies M = 0$. So ρ^* is faithful. Since $\mathfrak{A}^\circ$ is commutative, the map $|X|\, E_i \mapsto B_i^* = {}^t(\rho^*(|X|)E_i)$ gives an algebra isomorphism from $\mathfrak{A}^\circ$ to $\mathfrak{B}^*$. □

Proposition 3.14

(1) $P\, {}^tB_i P^{-1} = \mathrm{diag}(P_i(0), P_i(1), \ldots, P_i(d))$;

(2) $Q\, {}^tB_i^* Q^{-1} = \mathrm{diag}(Q_i(0), Q_i(1), \ldots, Q_i(d))$.

Proof. (1) By Theorem 3.3(6)

$$\begin{aligned}
&(\mathrm{diag}(P_i(0), P_i(1), \ldots, P_i(d))P)(\gamma, j)\\
=&P_i(\gamma)P_j(\gamma)\\
=&\sum_{k=0}^{d} p_{ij}^k P_k(\gamma)\\
=&(P\, {}^tB_i)(\gamma, j),
\end{aligned}$$

i.e., $\mathrm{diag}(P_i(0), P_i(1), \ldots, P_i(d))P = P\, {}^tB_i$

(2) By Theorem 3.3(7)

$$\begin{aligned}
&(\operatorname{diag}(Q_i(0), Q_i(1), \ldots, Q_i(d))Q)(\gamma, j)\\
=&Q_i(\gamma)Q_j(\gamma)\\
=&\sum_{k=0}^{d} q_{ij}^k Q_k(\gamma)\\
=&(Q\,{}^tB_i^*)(\gamma, j),
\end{aligned}$$

i.e., $\operatorname{diag}(Q_i(0), Q_i(1), \ldots, Q_i(d))Q = Q\,{}^tB_i^*$ □

Remark 3.15

Let $\mathfrak{X}$ be any commutative association scheme, we can compute P, Q, p_{ij}^k, q_{ij}^k from any one of these four collection of parameters.

Now we consider the case of group association schemes. Let G be any finite group, $\mathfrak{X}(G) = (G, \{R_i\}_{0\leqslant i\leqslant d})$ be the group association scheme. Suppose the conjugacy classes are $C_0, C_1, \ldots, C_d$ and $\chi_0, \chi_1, \ldots, \chi_d$ are the irreducible characters of G. We have the character table $T = (\chi_i(C_j))_{0\leqslant i,j\leqslant d}$. Furthermore, we have $f_i := \chi_i(1)$ and $k_i = |C_i|$. Then the character table of the finite group G and the first eigenmatrix of $\mathfrak{X}$ are related as follows:

$$P = \operatorname{diag}\left(f_0^{-1}, f_1^{-1}, \ldots, f_d^{-1}\right) T \operatorname{diag}\left(k_0, k_1, \ldots, k_d\right).$$

Remark 3.16: Fusion algebra

Bose-Mesner algebra (character algebra) is related to fusion algebra in physics (conformal field theory). There is a Verlinde's formula in physics. What does it correspond to in Bose-Mesner algebra? $P_i(j)$ and p_{ij}^k? [Ban93, Ban03, Gan05]

4 Terwilliger Algebra (Subconstituent Algebra)

For a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$, we have the Bose-Mesner algebra

$$\mathfrak{A} = \langle A_0, A_1, \ldots, A_d\rangle = \langle E_0, E_1, \ldots, E_d\rangle.$$

Now we fix an element $x_0 \in X$, we will define an algebra $T = T(x_0) = \langle \mathfrak{A}, \mathfrak{A}^*\rangle \subset M_{|X|}(\mathbb{C})$.

Let $V = \mathbb{C}\,X = \mathbb{C}^X$ considered as column space. We consider $\Gamma_i(x_0) := \{x \in X : (x_0, x) \in R_i\}$ and define $V_i^* := \mathbb{C}\,\Gamma_i(x_0)$ to be the space spanned by $\Gamma_i(x_0) \subset X$. So $\dim V_i^* = k_i = |\Gamma_i(x_0)|$. Naturally we have $V = V_0^* \perp V_1^* \perp \cdots \perp V_d^*$. Let $E_i^* : V \to V_i^*$ be the orthogonal projection from V to V_i^*, that is

$$E_i^* = \begin{cases} 1 & \text{if } x = y \text{ and } (x_0, x) \in R_i;\\ 0 & \text{otherwise.}\end{cases}$$

So $E_i^* E_j^* = \delta_{ij} E_i^*$ and $I = E_0^* + E_1^* + \cdots + E_d^*$. We define

$$A_i^* = \sum_{\alpha=0}^{d} Q_i(\alpha) E_\alpha^*, \tag{4}$$

then A_i^* are diagonal matrices. If $(x_0, x) \in R_\alpha$, then

$$A_i^*(x, x) = \sum_{i=0}^{d} Q_i(\gamma) E_\gamma^*(x, x) = Q_i(\alpha) = |X|\, E_i(x_0, x). \tag{5}$$

Proposition 4.1

(1) $E_i^* = \frac{1}{|X|} \sum_{j=0}^{d} P_i(j) A_j^*$;

(2) $A_i^* A_j^* = \sum_{k=0}^{d} q_{ij}^k A_k^*$;

(3) $Q_i(0), Q_i(1), \ldots, Q_i(d)$ are the eigenvalues of A_i^*.

Proof. (1) We multiply P from the right on both sides of $(A_0^*, A_1^*, \ldots, A_d^*) = (E_0^*, E_1^*, \ldots, E_d^*)Q$. The formula follows since $QP = I$.

(2) Note that A_i^*,A_j and $A_i^* A_j*$ are diagonal matrices. By Eq. (5),

$$\begin{aligned}(A_i^* A_j^*)(x,x) &= A_i^*(x,x) A_j^*(x,x)\\ &= |X|\, E_i(x_0,x)\, |X|\, E_j(x_0,x)\\ &= ((|X|\, E_i) \circ (|X|\, E_j))\, (x_0,x)\\ &= \sum_{k=0}^{d} q_{ij}^k E_k(x_0,x)\\ &= \sum_{k=0}^{d} q_{ij}^k A_k^*(x,x).\end{aligned}$$

(3) A_i^* has diagonal entries $Q_i(0), Q_i(1), \ldots, Q_i(d)$. □

Definition 4.2: dual Bose-Mesner algebra

$\mathfrak{A}^* = \mathfrak{A}^*(x_0) = \langle A_0^*, A_1^*, \ldots, A_d^* \rangle$ is called the **dual Bose-Mesner algebra**.

Proposition 4.3

(1) $\mathfrak{A}^* = \langle E_0^*, E_1^*, \ldots, E_d^* \rangle$;

(2) The map $|X|\, E_i \mapsto A_i^*$ gives the algebra isomorphism $\mathfrak{A}^\circ \to \mathfrak{A}^*$;

(3) The map $A_i^* \mapsto B_i^*$ gives the algebra isomorphism $\mathfrak{A}^* \to \mathfrak{B}^*$.

Proof. (1) Note that $(A_0^*, A_1^*, \ldots, A_d^*) = (E_0^*, E_1^*, \ldots, E_d^*)Q$ and the second eigenmatrix Q is invertible, hence $E_0^*, E_1^*, \ldots, E_d^*$ is also a basis of $\mathfrak{A}^*$.

(2) Proposition 4.1 tells us that $A_i^* A_j^* = \sum_{k=0}^{d} q_{ij}^k A_k^*$ and we also have $(|X|\, E_i) \circ (|X|\, E_j) = \sum_{k=0}^{d} q_{ij}^k (|X|\, E_k)$, so $|X|\, E_i \mapsto A_i^*$ gives the correspondence between the basis of the algebra $\mathfrak{A}^\circ$ and $\mathfrak{A}^*$.

(3) Theorem 3.13 tell us that the map $|X|\, E_i \mapsto B_i^*$ gives the algebra isomorphism from $\mathfrak{A}^\circ$ to $\mathfrak{B}^*$, while we also have that the map $|X|\, E_i \mapsto A_i^*$ gives the algebra isomorphism $\mathfrak{A}^\circ \to \mathfrak{A}^*$. Thus the map $A_i^* \mapsto B_i^*$ gives the algebra isomorphism $\mathfrak{A}^* \to \mathfrak{B}^*$. □

Definition 4.4: Terwilliger algebra

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative association scheme and $x_0 \in X$ fixed. $T = T(x_0) = \langle \mathfrak{A}, \mathfrak{A}^* \rangle \subset M_X(\mathbb{C})$ is called **Terwilliger algebra** and x_0 is called the **base point** of X.

Let $A, B \in M_X(\mathbb{C})$, we define the inner product of A and B by

$$(A, B) \coloneqq \tau\left(A \circ \overline{B}\right)$$

where $\tau(M) = \sum_{(x,y)\in X\times X} M(x,y)$. Equivalently $(A,B) = \operatorname{tr}\left(A\,\overline{{}^tB}\right)$.

Lemma 4.5

Let $\alpha, \beta, \gamma, i, j, k \in \{0, 1, \ldots, d\}$.

(1) $\left(E_\alpha A^*_\beta E_\gamma, E_i A^*_j E_k\right) = \delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k} q^\gamma_{\alpha\beta} m_\gamma$;

(2) $\left(E^*_\alpha A_\beta E^*_\gamma, E^*_i A_j E^*_k\right) = \delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k} p^\gamma_{\alpha\beta} k_\gamma$.

Corollary 4.6

(1) $q^k_{ij} \geqslant 0$;

(2) $E_\alpha A^*_\beta E_\gamma = 0 \iff q^\gamma_{\alpha\beta} = 0$;

(3) $E^*_\alpha A_\beta E^*_\gamma = 0 \iff p^\gamma_{\alpha\beta} = 0$.

Proof of Lemma 4.5. (1) Note that $\overline{{}^tE_k} = E_k$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we have

$$\begin{aligned}
&\left(E_\alpha A^*_\beta E_\gamma, E_i A^*_j E_k\right) \\
=&\operatorname{tr}\left(E_\alpha A^*_\beta E_\gamma \overline{{}^tE_k}\,\overline{{}^tA^*_j}\,\overline{{}^tE_k}\right) \\
=&\operatorname{tr}\left(E_i E_\alpha A^*_\beta E_\gamma E_k \overline{{}^tA^*_j}\right) \\
=&\delta_{\alpha i}\delta_{\gamma k}\operatorname{tr}\left(E_\alpha A^*_\beta E_\gamma \overline{{}^tA^*_j}\right) \\
=&\delta_{\alpha i}\delta_{\gamma k}\sum_{x,y\in X} E_\alpha(x,y) A^*_\beta(y,y) E_\gamma(y,x) \overline{A^*_j}(x,x) \\
=&\delta_{\alpha i}\delta_{\gamma k}\sum_{x,y\in X} E_\alpha(x,y)\,|X|\,E_\beta(x_0,y)E_\gamma(y,x)\,|X|\,\overline{E_j}(x_0,x) \\
=&\delta_{\alpha i}\delta_{\gamma k}\,|X|^2\sum_{x,y\in X} E_\beta(x_0,y)\left(E_\gamma \circ {}^tE_\alpha\right)(y,x)\,|X|\,\overline{E_j}(x_0,x) \\
=&\delta_{\alpha i}\delta_{\gamma k}\,|X|^2\left(E_\beta\left(E_\gamma \circ {}^tE_\alpha\right)E_j\right)(x_0,x_0) \\
=&\delta_{\alpha i}\delta_{\gamma k}\,|X|\sum_{\gamma=0}^{d} q^\nu_{\hat\alpha\gamma}(E_\beta E_\nu E_j)(x_0,x_0) \\
=&\delta_{\alpha i}\delta_{\gamma k}\delta_{\beta j}\,|X|\,q^\beta_{\hat\alpha\gamma}E_\beta(x_0,x_0) \\
=&\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k}q^\beta_{\hat\alpha\gamma}Q_\beta(0) \\
=&\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k}q^\beta_{\hat\alpha\gamma}m_\beta \\
=&\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k}q^\gamma_{\alpha\beta}m_\gamma.
\end{aligned}$$

(2)

$$\begin{aligned}
&\left(E^*_\alpha A_\beta E^*_\gamma, E^*_i A_j E^*_k\right)\\
=&\delta_{\alpha i}\delta_{\gamma k}\operatorname{tr}\left(E^*_\alpha A_\beta E^*_\gamma\, {}^t\! A_j\right)\\
=&\delta_{\alpha i}\delta_{\gamma k}\sum_{x,y\in X} E^*_\alpha(x,x)A_\beta(x,y)E^*_\gamma(y,y)\, {}^t\! A_j(y,x)\\
=&\delta_{\alpha i}\delta_{\gamma k}\sum_{x\in\Gamma_\alpha(x_0)}\sum_{y\in\Gamma_\gamma(x_0)} A_\beta(x,y)\, {}^t\! A_j(y,x)\\
=&\delta_{\alpha i}\delta_{\beta j}\delta_{\gamma k}p^\gamma_{\alpha\beta}k_\gamma.
\end{aligned}$$

□

Proposition 4.7

Let W be a subspace of V. If W is T-invariant, then $W^\perp$ is T-invariant.

Proof. Note that T is closed under transpose-complex conjugate. Take $v \in W^\perp$. For every $A \in T$ and $w \in W$, we have

$$\langle w, Av\rangle = \left\langle {}^t\!\bar{A}w, v\right\rangle = 0,$$

since ${}^t\!\bar{A}w \in W$. □

Lemma 4.8

(1) $E^*_i\mathbf{1} = A_i x_0$;

(2) $E_i x_0 = \frac{1}{|X|}A^*_i\mathbf{1}$;

(3) $\mathfrak{A}\, x_0 = \mathfrak{A}^*\,\mathbf{1}$.

Proof. (1) $E^*_i\mathbf{1} = \sum_{x\in\Gamma_i(x_0)} x = A_i x_0$.

(2) By Eq. (4), we have $E_i x_0 = \frac{1}{|X|}\sum_{\alpha=0}^d Q_i(\alpha)A_\alpha x_0 = \frac{1}{|X|}\sum_{\alpha=0}^d Q_i(\alpha)E^*_\alpha\mathbf{1} = \frac{1}{|X|}A^*_i\mathbf{1}$.

(3) Note that $\dim(\mathfrak{A}\, x_0) = \dim(\mathfrak{A}^*\,\mathbf{1}) = d+1$.

□

Definition 4.9

We call $\mathfrak{A}\, x_0 = \mathfrak{A}^*\,\mathbf{1}$ the **principal T-modules** (**primary T-module**).

Definition 4.10

(1) Define $v_i = E^*_i\mathbf{1}$ for $i \in \{0,1,\ldots,d\}$. The set of vectors $\{v_0,\ v_1,\ldots,\ v_d\}$ are called the **standard basis** of $\mathfrak{A}\, x_0 = \mathfrak{A}^*\,\mathbf{1}$. Note that:

$v_i = E^*_i\mathbf{1} = A_i x_0 \in v^*_i$;

$v_0 + v_1 + \cdots + v_d = \mathbf{1} \in V_0$.

(2) Define $v^*_i = E_i x_0$ for $i \in \{0,1,\ldots,d\}$. The set of vectors $\{v^*_0, v^*_1,\ldots,v^*_d\}$ are called the **dual standard basis** of $\mathfrak{A}\, x_0 = \mathfrak{A}^*\,\mathbf{1}$. Note that:

$v^*_i = E_i x_0 = \frac{1}{|X|}A^*_i\mathbf{1} \in V_i$;

$v^*_0 + v^*_1 + \cdots + v^*_d = x_0 \in V^*_0$.

Proposition 4.11

For principal T-modules $\mathfrak{A}\,x_0$, we get:

(1) $\dim(E_i\,\mathfrak{A}\,x_0) = 1$, for $i \in \{0, 1, \ldots, d\}$;

(2) $\dim(E_i^*\,\mathfrak{A}\,x_0) = 1$, for $i \in \{0, 1, \ldots, d\}$;

(3) $A_j v_i = \sum_{k=0}^{d} p_{ji}^k v_k$, representation of A_j with respect to the standard basis is B_j, where $B_j = (p_{ji}^k)$;

(4) $A_j^* v_i^* = \sum_{k=0}^{d} q_{ji}^k v_k^*$, representation of A_j with respect to the standard basis is B_j^*, where $B_j^* = (q_{ji}^k)$.

$\mathfrak{A}\,x_0$ has all the information on p_{ij}^k and q_{ij}^k.

$$v_j = A_j x_0 = \sum_{i=0}^{d} P_j(i) E_i x_0 = \sum_{i=0}^{d} P_j(i) v_i^*;$$

$$v_j^* = \frac{1}{|X|} A_j^* I = \frac{1}{|X|} \sum_{i=0}^{d} Q_j(i) E_i^* I = \frac{1}{|X|} \sum_{i=0}^{d} Q_j(i) v_i.$$

Problem 4.1

We want to decompose V into irreducible T-modules.

There are already many works.

(1) Terwilliger [Ter92, Ter93a, Ter93b] introduced the subconstituent algebra $\boldsymbol{T}$ (or Terwilliger algebra) of a commutative association scheme. He determined the structure of a thin, irreducible T-module of a P- and Q- polynomial association scheme.

(2) Tomiyama–Yamazaki [TY94] studied the Terwilliger algebra T of strongly regular graphs and determined the dimension of T.

(3) Bannai–Munemasa [TY94] studied the structure of the Terwilliger algebra of a group association scheme. They calculated an upper bound as well as a lower bound of the dimension of T.

(4) Tanabe [Tan97] determined all irreducible T-modules of Doob schemes.

(5) Go [Go02] considered the irreducible complex T-modules of hypercube and the multiplicities of each of the irreducible T-modules in $\mathbb{C}X$ are computed. Moreover, Go–Terwilliger [GT02] studied irreducible T-modules of distance regular graphs.

(6) Ito and his students are studying the irreducible T-modules of $J(v, k)$ and partial result was included in Liang–Tan–Ito [LTI17].

5 Related Concepts of Association Scheme

5.1 Duality of association scheme

Definition 5.1: Duality map

For a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ with the Bose-Mesner algebra $\mathfrak{A}$. A linear automorphism map $\Psi : \mathfrak{A} \mapsto \mathfrak{A}$ is called a **duality map**, if the following conditions are satisfied:

(1) $\Psi(MN) = \Psi(m) \circ \Psi(N)$ for all $M, N \in \mathfrak{A}$;

(2) $\Psi^2(M) = |X|\ {}^tM$ for all $M \in \mathfrak{A}$.

A commutative association scheme $\mathfrak{X}$ is called **self-dual** if there exists a duality map from $\mathfrak{A}$ to $\mathfrak{A}$.

The definition of "duality" comes from [BBJ97].

Example 5.2

(1) Hamming association scheme $H(d, q)$ is self-dual;

(2) Johnson association scheme $J(v, d)$ is not self-dual.

Proposition 5.3

Let Ψ be a duality map. Then

(1) ${}^t\Psi(M) = \Psi({}^tM)$;

(2) $\Psi(M \circ N) = \frac{1}{|X|}\Psi(M)\Psi(N)$.

Proof. (1) Note that

$$|X|\,\Psi({}^tM) = \Psi(|X|\ {}^tM) = \Psi(\Psi^2(M)) = \Psi^2(\Psi(M)) = |X|\ {}^t(\Psi(M)).$$

Thus,we have ${}^t\Psi(M) = \Psi({}^tM)$.

(2) Note that

$$|X|\ {}^t(\Psi(M)\Psi(N)) = \Psi^2(\Psi(M)\Psi(N)) = \Psi(\Psi^2(M) \circ \Psi^2(N)) = |X|^2\ {}^t\Psi(M \circ N).$$

Thus,we have $\Psi(M \circ N) = \frac{1}{|X|}\Psi(M)\Psi(N)$.

□

Proposition 5.4

The following are equivalent:

(1) $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is self-dual;

(2) By arranging the order of $R_0, R_1, \ldots, R_d$ appropriately, we get $P = \overline{Q}$.

Proof. (1) $\Rightarrow$ (2). Let Ψ be a duality map. By Proposition 5.3, $\Psi(M) = \Psi(J \circ M) = \frac{1}{|X|}\Psi(J)\Psi(M) = \Psi(E_0)\Psi(M)$ for all $M \in \mathfrak{A}$. Hence $\Psi(E_0) = I = A_0$. Moreover, $\Psi(E_i) = \Psi(E_i^2) = \Psi(E_i) \circ \Psi(E_i)$ and then entries of $\Psi(E_i)$ are either 0 or 1.

Since Ψ is the automorphism $\mathfrak{A} \mapsto \mathfrak{A}$, we have $\{\Psi(E_0), \Psi(E_1), \ldots, \Psi(E_d)\} = \{A_0, A_1, \ldots, A_d\}$. Changing the order of $R_0, R_1, \dots, R_d$ if necessary,we may assume that $\Psi(E_i) = A_i$ for all $i \in \{0, 1, \dots, d\}$. Then

$$\begin{aligned} A_i =& \Psi(E_i) = \frac{1}{|X|}\sum_{j=0}^{d} Q_i(j)\Psi(A_j) \\ =& \frac{1}{|X|}\sum_{j=0}^{d} Q_i(j)\Psi^2(E_j) \\ =& \sum_{j=0}^{d} Q_i(j)\,{}^tE_j = \sum_{j=0}^{d} Q_i(j)\overline{E_j}. \end{aligned}$$

So $A_i = \overline{A_i} = \sum_{j=0}^{d} \overline{Q_i(j)} E_j$. Thus,$P_i(j) = \overline{Q_i(j)}$.

(2) $\Rightarrow$ (1). Order $R_0, R_1, \dots, R_d$, so that $P = \overline{Q}$ i.e., $P_i(j) = \overline{Q_i(j)}$ for all i, j. Then it is easily checked that

$$\Psi(E_i) = A_i, \quad i \in \{0, 1, \dots, d\}$$

gives a self-dual map. □

Remark 5.5

There are some related works of Jaeger. [Jae96, JMN98]

5.2 Fusion scheme of association scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative association scheme. Combing R_i's, can we get a new association scheme?

Definition 5.6

Let $\{0, 1, \dots, d\}$ be the index set and $\{0, 1, \dots, d\} = \{0\} \sqcup \Lambda_1 \sqcup \Lambda_2 \sqcup \cdots \sqcup \Lambda_{\widetilde{d}}$.
Also,let $\widetilde{R}_0 = R_0$, $\widetilde{R}_1 = \cup_{j \in \Lambda_1} R_j, \dots, \widetilde{R}_{\widetilde{d}} = \cup_{j \in \Lambda_{\widetilde{d}}} R_j$. If $\widetilde{\mathfrak{X}} = (X, \{\widetilde{R}_i\}_{0 \leqslant i \leqslant \widetilde{d}})$ is an association scheme, then $\widetilde{\mathfrak{X}}$ is called a **fusion scheme** of $\mathfrak{X}$.

- Fusion scheme of $H(d, q)$ is classified for $q \geq 4$. See [Muz92a]
- Fusion scheme of $J(v, d)$ is classified for some $v \geq f(d)$. See [Muz92b, Uch92]
- Fusion schemes of group association scheme: many work by Q. Xiang, H. Tanaka and Iwakata.

Question 5.1

Let G be any finite group. Find all fusion schemes of $\mathfrak{X}(G)$.

Theorem 5.7: Bannai-Muzycluk criterion [Ban91]

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutative association scheme. Let $\widetilde{\mathfrak{X}} = (X, \{\widetilde{R}_i\}_{0 \leqslant i \leqslant \widetilde{d}})$ be a fusion scheme of $\mathfrak{X}$ if and only if the following conditions are satisfied:

(1) For each $i \in \{1, 2, \dots, \widetilde{d}\}$, ${}^t\widetilde{R}_i \in \{\widetilde{R}_0, \widetilde{R}_1, \dots, \widetilde{R}_{\widetilde{d}}\}$;

(2) There is another partition of $\{0, 1, \dots, d\}$, such that $\{0, 1, \dots, d\} = \{0\} \sqcup F_1 \sqcup F_2 \sqcup \cdots \sqcup F_{\widetilde{d}}$ and if we define $\widetilde{E}_0 = E_0, \widetilde{E}_1 = \sum_{j \in F_1} E_j, \dots, \widetilde{E}_{\widetilde{d}} = \sum_{j \in F_{\widetilde{d}} E_j}$, then ${}^t\widetilde{E}_i \in \{\widetilde{E}_0, \widetilde{E}_1, \dots, \widetilde{E}_{\widetilde{d}}\}$;

(3) Let $P|_{F_k \times \Lambda_l}$ be the restriction of P to $F_k \times \Lambda_l$, then each row sum is constant. That is, for each $i \in F_k$, $\sum_{j \in \Lambda_l} P_j(i)$ does not depend on i.

Proof. For each $\ell \in \{0, \ldots, \widetilde{d}\}$, define $\widetilde{A}_\ell = \sum_{j\in\Lambda_\ell} A_j$.

"⇒" Suppose $\widetilde{\mathfrak{X}}$ is a fusion scheme. Then $A_0 = \widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_{\widetilde{d}}$ are the adjacent matrices of $\widetilde{\mathfrak{X}}$. So (1) is obviously true.

Let $E_0 = \widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{d}}$ be the basis of primitive idempotents of $\widetilde{\mathfrak{X}}$. Then $\widetilde{E}_k$'s are mutually orthogonal primitive idempotents i.e., $\widetilde{E}_i\widetilde{E}_j = \delta_{ij}\widetilde{E}_i$. So (2) holds.

Let $\widetilde{P}$ be the first eigenmatrix of $\widetilde{\mathfrak{X}}$, then $\widetilde{A}_\ell\widetilde{E}_k = \widetilde{P}_\ell(k)\widetilde{E}_k$. So

$$\begin{aligned}(\sum_{j\in\Lambda_\ell} A_j)(\sum_{i\in F_k} E_i) &= \sum_{i\in F_k}\sum_{j\in\Lambda_\ell} A_jE_i \\ &= \sum_{i\in F_k}\big(\sum_{j\in\Lambda_\ell} P_j(i)\big)E_i \\ &= \widetilde{P}_\ell(k)\sum_{i\in F_k} E_i.\end{aligned}$$

So for all $i \in F_k$, $\sum_{j\in\Lambda_\ell} P_j(i) = \widetilde{P}_l(k)$. Thus (3) holds.

"⇐" Suppose the conditions (1) to (3) hold.

Since $\widetilde{\mathfrak{X}}$ is an association scheme, we have the conditions (1′) to (4′) Note that $\{\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_{\widetilde{d}}\}$ and $\{\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{d}}\}$ are linear independent. According to condition (3), we have

$$\begin{aligned}\widetilde{A}_\ell = \sum_{j\in\Lambda_\ell} A_j &= \sum_{j\in\Lambda_\ell}\sum_{v=0}^{d} P_j(v)E_v \\ &= \sum_{k=0}^{\widetilde{d}}\sum_{v\in F_k}\sum_{j\in\Lambda_\ell} P_j(v)E_v \\ &= \sum_{k=0}^{\widetilde{d}}\big(\sum_{v\in F_k}\widetilde{P}_\ell(v)E_v\big) = \sum_{k=0}^{\widetilde{d}}\widetilde{P}_\ell(k)\widetilde{E}_k.\end{aligned}$$

So the space spanned by $\{\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_{\widetilde{d}}\}$ and $\{\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{d}}\}$ are the same one. The space spanned by $\{\widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{d}}\}$ is closed under ordinary matrix multiplication. Therefore $\widetilde{A}_k\widetilde{A}_\ell \in< \widetilde{E}_0, \widetilde{E}_1, \ldots, \widetilde{E}_{\widetilde{d}} >=< \widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_{\widetilde{d}} >$, for all $k, \ell \in \{0, 1, \ldots, d\}$. According to the definition, the conclusion is obvious. □

5.3 Primitive association scheme and imprimitive association scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a commutative association scheme (not necessarily symmetric). Let (X, R_i) be a graph (maybe directed). The graph (X, R_i) is **connected** if for all $x, y \in X$, there exists a path $x = x_0, x_1, \ldots, x_r = y$ such that $(x_j, x_{j+1}) \in R_i$, for all $j \in \{0, 1, \ldots, r-1\}$. Since for every R_i there is a path of length 0 from x to x, we assume that x and x are connected by every R_i.

Definition 5.8: Primitive association scheme

An association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is called **primitive** if the graphs (X, R_i) are connected for all $i \in \{1, \ldots, d\}$. An association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is called **imprimitive**, if it is not primitive.

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be an association scheme. Define

$$\Gamma^{(i)}(x) := \{y \in X : \text{there exists a path from } x \text{ to } y \text{ in } (X, R_i)\}.$$

Proposition 5.9

For all $i \in \{0, 1, \ldots, d\}$,

(1) $|\Gamma^{(i)}(x)|$ does not depend on $x \in X$;

(2) $y \in \Gamma^{(i)}(x)$ implies $x \in \Gamma^{(i)}(y)$.

Proof. (1) Let $\Lambda_0 = \{0\}$, $\Lambda_1 = \{i\}$. For $\ell \in \{2, \ldots, d\}$, define

$$\Lambda_\ell := \{\nu : \text{there exists } \mu \in \Lambda_{\ell-1} \text{ such that } p^{\nu}_{\mu i} > 0 \text{ and } \nu \notin \Lambda_j \text{ for every } 0 \leqslant j \leqslant l-1\}.$$

Then $|\Gamma^{(i)}(x)| = \sum_{\ell=0}^{d} \left(\sum_{\nu \in \Lambda_\ell} k_\nu \right)$.

(2) Let $z \in \Gamma^{(i)}(y)$. Since $y \in \Gamma^{(i)}(x)$, we obtain $z \in \Gamma^{(i)}(x)$. And then $\Gamma^{(i)}(y) \subseteq \Gamma^{(i)}(x)$. By (1), $|\Gamma^{(i)}(y)| = |\Gamma^{(i)}(x)|$. Then,we have $\Gamma^{(i)}(y) = \Gamma^{(i)}(x)$. Note that $x \in \Gamma^{(i)}(x)$. So $x \in \Gamma^{(i)}(y)$.

□

Definition 5.10: Distribution graph

Given an association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$, define a graph Δ_{A_i} for each $i \in \{0, 1, \ldots, d\}$ by setting the vertex set $\{0, 1, \ldots, d\}$ and the arc set $\{(j, k) : p^k_{ij} > 0\}$. We call the graph Δ_{A_i} a **distribution graph** of $\mathfrak{X}$.

Note that if $\mathfrak{X}$ is symmetric then Δ_{A_i} is an undirected graph. Indeed, we have $k_\ell p^k_{ij} = k_j p^j_{i'\ell}$ and by symmetric $i' = i$.

Lemma 5.11

Let $i, \nu, \mu \in \{0, 1, \ldots, d\}$. Then the following conditions are equivalent:

(1) For every $(x_0, y) \in R_\mu$, there exists a vertex $x \in \Gamma_\nu(x_0)$ and a path in (X, R_i) from x to y;

(2) For a particular $(x_0, y) \in R_\mu$, there exists a vertex $x \in \Gamma_\nu(x_0)$ and a path in (X, R_i) from x to y;

(3) There is a path from ν to μ in Δ_{A_i}.

Proof. (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) There is a path $x = \mu_0, \mu_1, \ldots, \mu_r = y$ with $\nu_0 = \nu$, $\nu_r = \mu$, $(x_0, u_j) \in R_{\nu_j}$ where $j \in \{1, \ldots, r\}$. Then we have:

$$\begin{aligned} &p^{\nu_1}_{i\nu} = p^{\nu_1}_{\nu i} > 0 \\ &p^{\nu_2}_{i\nu_1} = p^{\nu_2}_{\nu_1 i} > 0 \\ &\cdots \\ &p^{\nu_r}_{i\nu_{r-1}} = p^{\mu}_{i\nu_{r-1}} > 0 \end{aligned}$$

So there exists a path from ν to μ in Δ_{A_i}.

(3) $\Rightarrow$ (1) By condition (3), there are positive numbers:

$$p^{\nu_1}_{i\nu} = p^{\nu_1}_{\nu i}, p^{\nu_2}_{i\nu_1}, \ldots,\ p^{\nu_r}_{i\nu_{r-1}} = p^{\mu}_{i\nu_{r-1}}$$

Namely, there is a path in (X, R_i) for any $(x_0, y) \in R_\mu$:

$$y = \mu_r,\ \mu_{r-1},\ \ldots,\ \mu_0 = x$$

such that $(x_0, y_{j-1}) \in R_{v_{j-1}}$ and $(\mu_{j-1}, \mu_j) \in R_i$. Then $(x_0, x) \in R_\nu$ and $x = \mu_0, \mu_1, \ldots, \mu_r = y$ give a path from x to y in (X, R_i). □

Corollary 5.12

In the distribution graph Δ_{A_i}. There is a path from ν to μ if and only if there is a path from μ to ν.

For $\nu, \mu \in \{0, 1, \ldots, d\}$, write $\nu \sim_i \mu$ if there exists a path from ν to μ in Δ_{A_i}. Then $\sim_i$ is a equivalent relation.

Proposition 5.13

Let Ω be the connected component of Δ_{A_i} containing 0. Let $R_\Omega := \bigcup_{\alpha\in\Omega} R_\alpha$. Then

(1) For every $x, y \in X$, $(x, y) \in R_\Omega$. $\Leftrightarrow$ There exists a path from x to y in (X, R_i).

(2) R_Ω is an equivalent condition on X.

Proof. (1) "$\Leftarrow$" Let $(x, y) \in R_\mu$ and there is a path from x to y in (X, R_i). Let $\nu = 0$ and $x_0 = x$. Then there is a path from 0 to μ in Δ_{A_i}. So $\mu \in \Omega$.

"$\Rightarrow$" Let $(x, y) \in R_\Omega$. Then there exists $\mu \in \Omega$, such that $(x, y) \in R_\mu$. So there is a path from 0 to μ in Δ_{A_i}. Put $\nu = 0$ and $x_0 = x$ in Lemma 5.11. Then there is a path from x to y in (X, R_i);

(2) Define the relation $\underset{R_i}{\sim}$ on X. $x \underset{R_i}{\sim} y$. $\Leftrightarrow$ There is a path from x to y in (X, R_i). This is an equivalent relation. Note that by (1), this is equivalent to $(x, y) \in R_\Omega$. So R_Ω is an equivalent relation.

□

Corollary 5.14

(X, R_i) is connected if and only if Δ_{A_i} is connected.

Proof. Because of the fact that:

(1) (X, R_i) is connected. $\Longleftrightarrow x \underset{R_i}{\sim} y$ for all $x, y \in X$;

(2) For $(x, y) \in R_\mu$, $x \underset{R_i}{\sim} y \Leftrightarrow \mu \in \Omega$;

we have

$$\begin{aligned}(X, R_i) \text{ is connected.} &\Longleftrightarrow \Omega = \{0, 1, \ldots, d\} \\ &\Longleftrightarrow \Delta_{A_i} \text{ is connected.}\end{aligned}$$

□

Proposition 5.15

The following are equivalent:

(1) $\mathfrak{X}$ is imprimitive;

(2) There exists a subset Ω with $0 \subsetneqq \Omega \subsetneqq \{0, 1, \ldots, d\}$ such that $R_\Omega = \bigcup_{\alpha\in\Omega} R_\alpha$ gives an equivalent relation on X.

Proof. (1) $\Rightarrow$ (2) there exists $i \neq 0$, $i \in \{0, 1, \ldots, d\}$ such that (X, R_i) is disconnected. By Corollary 5.14, Δ_{A_i} is disconnected. Let Ω be the connected component of $\{0, 1, \ldots, d\}$ which contains 0. Then R_Ω satisfies the condition (2).

(2) $\Rightarrow$ (1) Let $i \in \Omega$ with $i \neq 0$. $x \underset{R_i}{\sim} y$ implies x and y are in the same equivalent class in R_Ω. So (X, R_i) is disconnected. □

Define a matrix $B_i^* := (q_{ij}^k)_{0\leqslant j\leqslant d, 0\leqslant k\leqslant d}$. The eigenvalues of B_i^* are $Q_i(0), Q_i(1), \ldots, Q_i(d)$. Define a matrix $B_i := (p_{ij}^k)_{0\leqslant j\leqslant d, 0\leqslant k\leqslant d}$. The eigenvalues of B_i are $P_i(0), P_i(1), \ldots, P_i(d)$.

If $\mathfrak{X}$ is symmetric, then $\forall Q_i(j) \in \mathbb{R}$. $\sum_{j=0}^d q_{ij}^k = m_i = Q_i(0)$ and $P_i(0) = k_i$.

Definition 5.16: Representation graph

Given an association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$, define a graph Δ_{E_i} for each $i \in \{0, 1, \ldots, d\}$ by setting the vertex set $\{0, 1, \ldots, d\}$ and the arc set $\{(j, k) : q_{ij}^k > 0\}$. We call the graph Δ_{E_i} a **representation graph** of $\mathfrak{X}$.

Note that if $\mathfrak{X}$ is symmetric then Δ_{E_i} is also an undirected graph.

Proposition 5.17

$\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is a commutative association scheme.

(1) $|P_j(i)| \leqslant P_j(0) = k_j$;

(2) The multiplicity of the eigenvalue $P_j(0)$ in B_j is 1. $\Leftrightarrow$ the distribution graph Δ_{A_j} is connected;

(3) $|Q_j(i)| \leqslant Q_j(0) = m_j$;

(4) The multiplicity of the eigenvalue $Q_j(0)$ in B_j^* is 1. $\Leftrightarrow$ the representation graph Δ_{E_j} is connected.

Proof. (1) We have the following equation:

$$(x_0, x_1, \ldots, x_d)B_j = P_j(i)(x_0, x_1, \ldots, x_d).$$

Let $j_0 \in \{0, 1, \ldots, d\}$ with $|x_{j_0}| \leqslant |x_j|$. Consider the j_0-th coordinate, then we have:

$$\sum_{\ell=0}^{d} B_j(\ell, j_0)x_l = P_j(i)x_{j_0}$$

So

$$|P_j(i)x_{j_0}| \leqslant \sum_{\ell=0}^{d} |x_l|B_j(\ell, j_0) \leqslant |x_{j_0}| \sum_{\ell=0}^{d} P_{j\ell}^{j_0} = |x_{j_0}|k_j.$$

which means $|P_j(i)| \leqslant k_j$. Thus (1) holds.

(2) (2) is similar to (4).

(3) (3) is similar to (1) and we can use $B_j^*(\ell, j_0) = q_{j\ell}^{j_0}$ and $\sum_{\ell=0}^d q_{j\ell}^{j_0} = m_j$.

(4) Without loss of generality, we take $j = 1$.

"$\Leftarrow$" Suppose Δ_{E_1} is connected, then the dimension of the space of the eigenvalue $Q_i(0)(= m_i)$ is 1. We have the following equation:

$$(x_0, x_1, \ldots, x_d)B_1^* = m_1(x_0, x_1, \ldots, x_d)$$

with $m_1 = Q_1(0)$.

Let $i_0 \in \{0, 1, \ldots, d\}$ with $|x_{i_0}| \geq |x_l|$ $(0 \leqslant l \leqslant d)$. (We can assume that $x_{i_0} > 0$.) Consider the i_0-th coordinate of $(x_0, x_1, \ldots, x_d)B_1^* = m_1(x_0, x_1, \ldots, x_d)$ with $m_1 = Q_1(0)$, then we have:

$$m_1 x_{i_0} = \sum_{k=0}^{d} x_k q_{1k}^{i_0} = \sum_{k=0}^{s} x_{v_k} q_{1k}^{i_0} \leqslant x_{i_0} \sum_{k=0}^{s} q1{v_k}^{i_0} = x_{i_0} m_1.$$

So $x_{v_1} = x_{V_2} = \cdots = x_{v_s} = x_{i_0}$ and as a result Δ_{E_1} is connected.

"⇒" If Δ_{E_1} is not connected and it has r connected components, the dimension of the eigenvalue space for $Q_1(0)$ is r.

□

Proposition 5.18

The following are equivalent:

(1) there exists $j \neq 0$, such that Δ_{A_j} is disconnected;

(2) there exists $i \neq 0$, such that Δ_{E_i} is disconnected.

Proof. (2) ⇒ (1) By Proposition 5.17 (4), the multiplicity of $Q_i(0)$ in B_i^* is greater than 1, i.e., there exists $j \neq 0$ such that $Q_i(j) = Q_i(0)$. Since the following equation is satisfied:

$$\frac{Q_j(i)}{m_j} = \frac{\overline{P_i(j)}}{k_i}$$

and

$$\frac{Q_i(j)}{m_i} = \frac{\overline{P_j(i)}}{k_j}.$$

Also, we have

$$m_i = Q_i(0), k_j = Q_j(0).$$

Then we have

$$1 = \frac{Q_i(j)}{Q_i(0)} = \frac{\overline{P_j(i)}}{P_j(0)}.$$

Hence $P_j(i) = P_j(0) = k_j$. By Proposition 5.17 (2), Δ_{A_j} is disconnected.

(1) ⇒ (2) Similar. □

Proposition 5.19

The followings are equivalent:

(1) $\mathfrak{X}$ is imprimitive;

(2) Δ_{A_j} is connected for every $j \neq 0$;

(3) Δ_{E_i} is connected for every $i \neq 0$.

Let $\Omega\Omega' \subset \{0, 1, \ldots, d\}$. We introduce the following notations.

- $A_\Omega = \sum_{\alpha\in\Omega} A_\alpha$;
- $R_\Omega = \sum_{\alpha\in\Omega} R_\alpha$;
- $\Omega\Omega' = \{k :$ exists $i \in \Omega, \exists j \in \Omega'$ such that $p_{ij}^k \neq 0\}$.

Proposition 5.20

(1) $\Omega\Omega' = \{k : (A_\Omega \circ A_{\Omega'}) \circ A_k \neq 0\}$;

(2) $(\Omega\Omega')\Omega'' = \Omega(\Omega'\Omega'') = \{k : (A_\Omega \circ A_{\Omega'} \circ A_{\Omega''}) \circ A_k \neq 0\}$.

Lemma 5.21

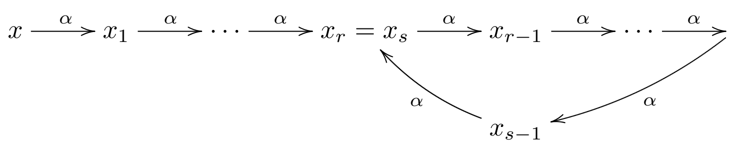
For $\Omega \subset \{0, 1, \ldots, d\}$, the following are equivalent:

(1) $\Omega^2 = \Omega$;

(2) R_Ω gives an equivalent relation on X.

Proof. (1) $\Rightarrow$ (2) Consider the graph (X, R_Ω). Take $(x, y) \in R_\alpha, \alpha \in \Omega$. Consider a path

$$x = x_0 \overset{\alpha}{\rightharpoondown} x_1 \overset{\alpha}{\rightharpoondown} \cdots \overset{\alpha}{\rightharpoondown} x_r = x_s \overset{\alpha}{\rightharpoondown} x_{r+1} \overset{\alpha}{\rightharpoondown} \cdots \overset{\alpha}{\rightharpoondown} x_{s-1} \overset{\alpha}{\rightharpoondown} x_s.$$



Since $\Omega = \Omega^2$, we have $(x_i, x_j) \in \Omega$, for all $i < j$. In particular $(x_r, x_r) = (x_r, x_s) \in R_\Omega$, then $0 \in \Omega$. And we have $(x_s, x_{s-1}) = (x_r, x_{s-1}) \in R_\Omega$, then $\alpha' \in \Omega$. This implies reflexivity and symmetricity. Note that $\Omega^2 = \Omega$ implies the transitivity.

(2) $\Rightarrow$ (1) R_Ω is an equivalence relation means, $(A_\Omega)^2 = k_\Omega A_\Omega$, where $k_\Omega = \sum_{\alpha \in \Omega} k_\alpha$. So we have $\Omega^2 = \Omega$. $\square$

5.4 Subscheme and quotient scheme

Recall

We say $G \to X$ is trancitively, which shows as follows

- G is called imprimitive if and only if there exists a partion of $X = X_1 \sqcup X_2 \sqcup \ldots \sqcup X_r$ such that G stabilizes $X_1, \ldots, X_r$ globally.
- G is primitive $\Leftrightarrow$ $G_x (x \in X)$ is a max subgraph of G, $G_x \underset{\neq}{\subsetneq} G_{X_1} \underset{\neq}{\subsetneq} G$.

According to Proposition 5.15 and Lemma 5.21, $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant s})$ is imprimitive commutative association scheme if and only if there exists $\Omega \subset \{0, 1, \ldots, d\}, \Omega \neq \{0\}, \Omega \neq \{0, 1, \ldots, d\}$, such that $\Omega^2 = \Omega$, and R_Ω gives an equivalence relation on X.

Let $X_1, X_2, \ldots, X_r$ be the decomposition of X into equivalence relation by Ω.

Then $\Sigma = \{X_1, X_2, \ldots, X_r\}$ is called the **system of imprimitivity**. Then $A_\alpha (\alpha \in \Omega)$ is in the following form

$$A_\alpha = \begin{array}{c} \\ X_0 \\ X_1 \\ \vdots \\ \vdots \\ X_r \end{array} \begin{array}{c} \begin{array}{ccccc} X_0 & X_1 & \cdots & \cdots & X_r \end{array} \\ \begin{pmatrix} * & 0 & \cdots & \cdots & 0 \\ 0 & * & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & * \end{pmatrix} \end{array}.$$

Note that

$$|X_1| = |X_2| = \cdots = |X_r| = \sum_{\alpha \in \Omega} K_\alpha = k_\Omega = \frac{|X|}{r},$$

we have

$$A_\Omega = \begin{array}{c} \\ X_0 \\ X_1 \\ \vdots \\ \vdots \\ X_r \end{array} \begin{array}{c} \begin{array}{ccccc} X_0 & X_1 & \cdots & \cdots & X_r \end{array} \\ \begin{pmatrix} J & 0 & \cdots & \cdots & 0 \\ 0 & J & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & J & 0 \\ 0 & \cdots & \cdots & 0 & J \end{pmatrix} \end{array}.$$

There are two association schemes naturally attached to this situation

- Subscheme: association scheme on each X_i.
- Quotient scheme: association scheme on Σ.

Take $Y = X_1$, then we have that $A_\alpha|_{Y\times Y}$ is an adjacent matrix of the graph (Y, R_α) of valency K_α. By $\Omega^2 = \Omega$, and $\Omega^2 = \{k : (A_\Omega \circ A_\Omega) \circ A_k \neq 0\}$. The space spanned by $\{A_\alpha : \alpha \in \Omega\}$ is closed under ordinary matrix multiply, i.e., $\mathfrak{A}_\Omega := \langle A_\alpha : \alpha \in \Omega\rangle$ is a subalgebra of $\mathfrak{A}$.

Since $0 \in \Omega$ and $A_0 = I \in \mathfrak{A}_\Omega$, we have $\dim\langle A_\alpha|_{Y\times Y} | \alpha \in \Omega\rangle = |\Omega| = \dim\langle A_\alpha | \alpha \in \Omega\rangle$.

Since $\langle A_\alpha|_{Y\times Y}\rangle$ contains $I = A_0|_{Y\times Y}$ and $J = A_\Omega|_{Y\times Y}$, it satisfies the conditions (1′) to (5′) of a commutive association scheme. So $(Y, \{R_\alpha|_{Y\times Y}\}_{\alpha\in\Omega}$ is a commutive association scheme and $\langle A_\alpha|_{Y\times Y} | \alpha \in \Omega\rangle$ is its Bose-Mesner algebra.

Definition 5.22: Subscheme

$\mathfrak{Y} = (Y, \{R_\alpha|_{Y\times Y}\}_{\alpha\in\Omega})$ is called a **subscheme** of $\mathfrak{X}$.

Note that $|\Omega| = s+1$. Then $\mathfrak{A}_\Omega$ has $s+1$ primitive idempotents (unique as a set) each of them is a sum of $E_0, E_1, \ldots, E_d$. Namely there is a partition, $\{0, 1, \ldots, d\} = \Lambda_0 \sqcup \Lambda_1 \sqcup \ldots \sqcup \Lambda_s$ with $\{E_{\Lambda_i} = \sum_{\alpha\in\Lambda_i} E_\alpha : 0 \leqslant i \leqslant s\}$ are the primitive idempotents of $\mathfrak{A}_\Omega$.

So $\{E_{\Lambda_i}|_{Y\times Y} : \alpha \in \Omega\}$ is a primitive idempotent basis of $\langle A_\alpha|_{Y\times Y} : \alpha \in \Omega\rangle$. Since

$$A_\Omega|_{Y\times Y} = J_{k_\Omega} \in \langle A_\alpha|_{Y\times Y} | \alpha \in \Omega\rangle,$$

we have

$$\frac{1}{k_\Omega} A_\Omega|_{Y\times Y} = \frac{1}{k_\Omega} J_{k_\Omega} = E_{\Lambda_i}|_{Y\times Y}$$

for some $i \in \{0, 1, \ldots, d\}$. By changing the ordering of

$$\{\Lambda_i | 0 \leqslant i \leqslant s\},$$

we have $E_{\Lambda_i}|_{Y\times Y} = \frac{1}{k_\Omega} J_{k_\Omega}$, let us denote $\Lambda = \Lambda_0$.

Proposition 5.23

We use the previous notation, for the subscheme $\mathfrak{Y}$ of a commutive association scheme $\mathfrak{X}$, the following conditions hold,

(1) $0 \in \Lambda$;

(2) $\Lambda \circ \Lambda := \{k : \text{exists } i, j \in \Lambda, \text{ such that } q_{ij}^k \neq 0\ \} = \Lambda$;

(3) The first eigenmatrix $\big(P_j(\Lambda_i)\big)_{0\leqslant i\leqslant s, j\in\Omega}$ satisfies $P_j(\Lambda_i) = P_j(\alpha)$ for all $\alpha \in \Lambda_i, j \in \Omega$;

(4) The second eigenmatrix $(Q_{\Lambda_j}(i))_{0\leqslant j\leqslant s, i\in\Omega}$ of $\mathfrak{Y}$ satisfies $Q_{\Lambda_j}(i) = \frac{|Y|}{|X|}\sum_{\alpha\in\Lambda_j} Q_\alpha(i)$. Moreover, if $i \in \Lambda$, then $\sum_{\alpha\in\Lambda_j} Q_\alpha(i) = 0$.

Proof. (1) $E_0E_\Lambda = \frac{1}{|X|}\frac{1}{k_\Omega}JA_\Omega = \frac{1}{|X|}J = E_0$, so $0 \in \Lambda$.

(2) $k_\Omega E_\Lambda = A_\Omega$, $A_\Omega \circ A_\Omega = A_\Omega$, $E_\Lambda \circ E_\Lambda = \frac{1}{|X|}\sum_{i,j\in\Lambda}(\sum_{k=0}^{d} q_{ij}^k E_k)$ so $\Lambda \circ \Lambda = \Lambda$.

(3) The correspondence $A_\alpha \mapsto A_\alpha|_{Y\times Y}$ gives an algebra isomorphism $\mathfrak{A}_\Omega$ to $\langle A_\alpha|_{Y\times Y}|\alpha \in \Omega\rangle$. Then $A_j = \sum_{i=0}^{s} P_j(\Lambda_i)E_{\Lambda_i}$, $A_j = \sum_{\alpha=0}^{d} P_j(\alpha)E_\alpha$, for $j \in \Omega$.

(4) For $j \in \{0, 1, \ldots, s\}$, we have $E_{\Lambda_j} = \frac{1}{|Y|}\sum_{i\in\Omega} Q_{\Lambda_j}(i)A_i$. While $E_\alpha = \frac{1}{|X|}\sum_{i=0}^{d} Q_\alpha(i)A_i$, since $E_{\Lambda_j} = \sum_{\alpha\in\Lambda_j} E_\alpha$.

□

Define $\mathfrak{A}_\Lambda := \langle E_i : i \in \Lambda\rangle \subset \mathfrak{A}$ with $|\Omega| = s+1$. Let $|\Lambda| = t+1$, then $\dim(\mathfrak{A}_\Lambda) = t+1$. Since $\Lambda \circ \Lambda = \Lambda$, $\mathfrak{A}_\Lambda$ is closed under Hadamard product.

Since $E_\Lambda = \frac{1}{k_\Omega}A_\Omega$ and ${}^tA_\Omega = A_\Omega$, for each $i \in \Lambda$, there exists $j \in \Lambda$, such that ${}^tE_i = E_j$. So $\mathfrak{A}_\Lambda$ is closed under taking transpose.

Since $0 \in \Lambda$ and $|X|E_0 = J \in \mathfrak{A}_\Lambda$, then $J \in \mathfrak{A}_\Lambda$ (identity with respect to Hadamard product).

For each $j \in \Lambda$, $E_\Lambda E_j = E_j E_\Lambda = E_j$, so E_Λ is the identity with respect to ordinary product in $\mathfrak{A}_\Lambda$. Moreover, $\mathfrak{A}_\Lambda$ has the basis of primitive idempotents, so there is a partition, $\{0, 1, \ldots, d\} = \Omega_0 \sqcup \Omega_1 \sqcup \ldots \sqcup \Omega_t$ with primitive idempotent basis with respect to Hadamard product

$$A_{\omega_i} = \sum_{\alpha\in\Omega_i} A_\alpha$$

for $0 \leqslant i \leqslant t$. Then $\mathfrak{A}_\Lambda = \langle A_\Omega : 0 \le i \le t\rangle$. Write Ω_0 with $0 \in \Omega_0$. Then

$$J = |X|E_0 = A\Omega_0 + A\Omega_1 + \cdots + A\Omega_t.$$

Proposition 5.24

The following hold.

(1) For each $i \in \{1, \ldots, t\}$, we have $A_{\Omega_i}A_\Omega = A_\Omega A_{\Omega_i} = k_\Omega A_\Omega$.

(2) For the system of imprimitivity $X_1, \ldots, X_r$, we have $A_{\Omega_i}|_{X_\alpha\times X_\beta} = 0$ or J.

(3) Let $\Sigma = \{X_1, X_2, \ldots, X_r\}$ and

$$D_i(\alpha,\beta) = \begin{cases} 1, & \text{if } A_{\Omega_i}|_{X_\alpha\times X_\beta} = J; \\ 0, & \text{if } A_{\Omega_i}|_{X_\alpha\times X_\beta} = 0. \end{cases}$$

Then $A_{\Omega_i} = D \otimes J$.

Proof. (1) $\mathfrak{A}_\Omega = \langle E_j : j \in \Lambda\rangle = \langle A_{\Omega_i} : 0 \leqslant i \leqslant t\rangle$ and $E_\Lambda = \frac{1}{k_\Omega}A_\Omega$ is the identity element of $\mathfrak{A}_\Lambda$.

(2) Compare the $X_\alpha \times X_\beta$ block of $A_\Omega A_{\Omega_i} = k_\Omega A_{\Omega_i}$. We have

$$J(A_\Omega)_{\alpha,\beta} = k_\Omega(A_\Omega)_{\alpha,\beta} \tag{6}$$

the entries of $(A_{\Omega_i})_{\alpha,\beta}$ are 0 and 1. So in order to have the equality in Eq. (6), either $(A_{\Omega_i})_{\alpha,\beta} = J$ or 0.

(3) It is obvious from (2).

□

Since each $E_j \in \mathfrak{A}_\Lambda$ is a linear combination of $\{A_{\Omega_\ell} : 0 \leqslant \ell \leqslant t\}$ we have

$$\begin{aligned} E_j &= \frac{1}{|X|} \sum_{\ell=0}^{t} \widetilde{Q}_j(\ell) A_{\Omega_\ell} \\ &= \frac{1}{r} (\sum_{\ell=0}^{t} \widetilde{Q}_j(\ell) D(\ell) \otimes (\frac{1}{k_\Omega} J). \end{aligned}$$

Define

$$F_j = \frac{1}{r} \sum_{l=0} t\widetilde{Q}_j(l) \cdot D(l),$$

where $j \in \{0, 1, \ldots, t\}$. Note that $|X| = rk_\Omega$.

Proposition 5.25

(1) Define the map φ from $\langle E_i : i \in \Lambda \rangle$ to $\langle D_i : 0 \leqslant i \leqslant t \rangle$ by $\varphi(\frac{1}{k_\Omega} A_{\Omega_i}) = D_i$. then φ gives the algebra isomorphism.

(2) For each i, $(0 \leqslant i \leqslant t)$, there exists $j \in \{0, 1, \ldots, t\}$ such that ${}^t D_i = D_j$.

(3) If $A_\Omega = A_{\Omega_0}$ then $D_0 = I, \Omega_0 = \Omega$.

(4) $\langle D_i : 0 \leqslant i \leqslant t \rangle$ is the Bose-Mesner algebra of a commutive association scheme on Σ (the system of imprimitivity).

(5) $F_0, F_1, \ldots, F_t$ become the primitive idempotent basis of Bose-Mesuer algebra $\langle D_i : 1 \leqslant i \leqslant t \rangle$, we also have $\varphi(E_j) = F_j$.

Proof. (1) Note that $(\frac{1}{k_\Omega} J)^2 = \frac{1}{k_\Omega} J$. So $\varphi : D_i \otimes (\frac{1}{k_\Omega} J) \mapsto D_i$ gives the isomorphism from

$$\mathfrak{A}_\Lambda = \langle A_{\Omega_i} \ 0 \leqslant i \leqslant t \rangle$$

to

$$\langle D_i : 0 \leqslant i \leqslant t \rangle \subset M_r(\mathbb{C}).$$

(2) Define $\Omega_i' = \{\alpha' : \alpha \in \Omega_i\}$, then ${}^t A_{\Omega_i} = A_{\Omega_i'}$, for $i \in \{0, 1, \ldots, t\}$. So Ω_Λ is closed under taking transpose. Then $A_{\Omega_i'} \in \mathfrak{A}_\Lambda$ and $\{A_{\Omega_0}, A_{\Omega_1}, \ldots, A_{\Omega_t}\}$ are idempotent with respect to Hadamard product of $\mathfrak{A}_\Lambda$. Hence $A_\Omega' \in \{A_{\Omega_0}, A_{\Omega_1}, \ldots, A_{\Omega_t}\}$ i.e., $\Omega_i' = \Omega_j$ for some j. That means for each $i \in \{0, \ldots mt\}$ there exist j such ${}^t D_i = D_j$.

(3) $A_\Omega = k_\Omega E_\Lambda \in \mathfrak{A}_\Lambda$, so A_Ω is a sum of $\{A_{\Omega_0}, A_{\Omega_1}, \ldots, A_{\Omega_t}\}$. Since $A_\Omega = I \otimes J$ and $A_\Omega = \sum_{\ell \in \Omega_0} A_l$, then $A_\Omega = A_{\Omega_0}$. Hence $\Omega_0 = \Omega$ and $D_0 = I$.

(4) By $J = A_{\Omega_0} + A_{\Omega_1} + \cdots + A_{\Omega_t}$, we have

$$D_0 + D_1 + \cdots + D_t = J.$$

so, all the condition (1′) to (5′) of Bose-Mesner algebra are satisfied.

(5) It is obvious from (1).

□

Definition 5.26: Quotient scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a commutive association scheme. Suppose $\mathfrak{X}$ is imprimitive i.e., there exists Ω such that $0 \subsetneqq \Omega \subsetneqq \{0, 1, \ldots, d\}$. By Proposition 5.25 (4), the commutive association scheme $(\Sigma, D_{i0 \leq i \leq t})$ is well-defined. And this is called the **quotient scheme** of $\mathfrak{X}$.

Proposition 5.27

The $\widetilde{P}$ and $\widetilde{Q}$ of the quotient scheme are

(1) For $i \in \Lambda$ and $0 \leqslant j \leqslant t$, we have

$$\widetilde{P}_j(i) = \frac{1}{k_\Omega} \sum_{\alpha \in \Omega_j} P_\alpha(i),$$

and for $i \notin \Lambda, 0 \leqslant j \leqslant t$ we have

$$\sum_{\alpha \in \Omega_j} P_\alpha(j) = 0.$$

(2) For every $\alpha \in \Omega, j \in \Lambda$,

$$\widetilde{Q}_j(i) = Q_j(\alpha).$$

Proof. In the quotient scheme,

$$D_j = \sum_{i \in \Lambda} \widetilde{P}_j(i) F_i, \qquad 0 \leqslant j \leqslant t$$

and

$$F_j = \sum_{i=0}^{t} \widetilde{Q}_j(i) D_i, \qquad j \in \Lambda.$$

We have defined the algebra isomorphism

$$\varphi : \langle E_i : i \in \Lambda \rangle \to \langle D_i : 0 \leqslant i \leqslant t \rangle,$$

then $\varphi(\frac{1}{k_\Omega} A_{\Omega_i}) = D_i$. If we pull back, in $\mathfrak{A}_\Lambda = \langle E_i : i \in \Lambda \rangle = \langle A_{\Omega_i} : 0 \leqslant i \leqslant t \rangle$, we have

$$\frac{1}{k_\Omega} A_{\Omega_j} = \sum_{i \in \Lambda} \widetilde{P}_j(i) E_i, \qquad 0 \leqslant j \leqslant t$$

and

$$E_j = \frac{1}{|X|} \sum_{i=0}^{t} \widetilde{Q}_j(i) A_{\Omega_i}, \qquad j \in \Lambda.$$

While

$$A_{\Omega_j} = \sum_{\alpha \in \Omega_j} A_\alpha = \sum_{i=0}^{d} \Big(\sum_{\alpha \in \Omega_j} P_\alpha(i) \Big) E_i$$

and

$$E_j = \frac{1}{|X|} \sum_{\alpha=0}^{d} Q_j(\alpha) A_\alpha.$$

Hence (1) and (2) follow. ∎

6 Distance Regular Graph and P-polynomial Association Scheme

6.1 Distance regular graph

Definition 6.1: Distance regular graph

Let $\Gamma = (X, E)$ be an undirected simple graph diameter d. We define $\Gamma_i(x) = \{y \in X : \mathrm{d}(x, y) = i\}$, where $\mathrm{d}(x, y)$ is the distance from x to y. Let $y \in \Gamma_i(x)$, if we have the following items

- $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i, \quad (i = 1, 2, \ldots, d)$;
- $|\Gamma_i(x) \cap \Gamma_1(y)| = a_i, \quad (i = 0, 1, \ldots, d)$;
- $|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i, \quad (i = 1, 2, \ldots, d-1)$;

are independent of $x \in X$ and $y \in \Gamma_i(x)$ (but depend only on i), then Γ is called a **distance regular graph**.

We usually assume $c_0 = 0, b_d = 0$ or not defined. Moreover, $b_0 = k$ is the valency of regular graph Γ.

One can check that:

- $k = c_i + a_i + b_i, (i = 0, 1, \ldots, d)$.
- $c_1 = |\Gamma_0(x) \cap \Gamma_1(y)| = |\{x\}|$ with $y \in \Gamma_1(y)$.
- $c_i > 0$, for $i = 1, 2, \ldots, d$, since Γ is of diameter d.
- $a_0 = 0$.

Remark 6.2

Let $k_i = |\Gamma_i(x)|$, then $k_0 = 1, k_1 = k, k_{i+1} = k_i \cdot \frac{b_i}{c_{i+1}}$. In fact, we can count the number of edges between $\Gamma_i(x)$ and $\Gamma_{i+1}(x)$.

Proposition 6.3

Let $\Gamma = (X, E)$ be a distance regular graph, define $R_i = \{(x, y) \in X \times X : d(x, y) = i\}$, then $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d}$ is a symmetric association scheme of class d.

Proof. Obviously, A_i's satisfy the conditions (1′) to (3′), (5′) and (6′).

Here we just show the proof of (4′). Since we have the following equations

$$A_j A_1(x, y) = \sum_{z \in X} A_j(x, z) A_i(z, y) = |\Gamma_j(x) \cap \Gamma_1(y)| = \begin{cases} c_{j+1}, & \text{if } d(x, y) = j + 1; \\ a_j, & \text{if } d(x, y) = j; \\ b_{j-1}, & \text{if } d(x, y) = j - 1\ ; \\ 0, & \text{if } |d(x, y) - j| \geqslant 2. \end{cases} \tag{7}$$

Then for $1 \leqslant j \leqslant d - 1$, we have $A_j A_1 = c_{j+1} A_{j+1} + a_j A_j + b_{j-1} A_{j-1}$. Hence A_j is a polynomial of degree j of A_1. Since the graph Γ has valency $k = b_0$,

$$(A_1 - kI)J = A_1 J - kJ = 0,$$

thus

$$(A_1 - kI)(A_0 + A_1 + \cdots + A_d) = 0,$$

where $A_0, A_1, \dots, A_d$ are linear independent. So the minimum polynomial of A_1 is given by

$$(x-k)(\text{polynomial of degree } d \text{ on } X)$$

The vector space spanned by $A_0, A_1, \dots, A_d$ is closed under ordinary matrix multiplication, i.e., there exists $p_{ij}^k \in \mathbb{Z}_{\geqslant 0}$ such that $A_iA_j = \sum_{l=0}^{d} p_{ij}^l A_l$, so (4′) holds. Hence $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is a symmetric association scheme. □

6.2 P-polynomial association scheme

Definition 6.4: P-polynomial association scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a symmetric association scheme of class d, choosing some ordering of $R_0, R_1, \dots, R_d$, if each $A_i (i = 1, 2, \dots, d)$ is a polynomial of degree i on A_1 (i.e., $A_i = v_i(A_1)$, for some polynomial v_i of degree i. $\mathfrak{X}$ is called a **P-polynomial association scheme**.

Note that the Bose-Mesner algebra of $\mathfrak{X}$ is generated by A_1.

Theorem 6.5

(1) Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a P-polynomial association scheme (with respect to some ordering of $R_0, R_1, \dots, R_d$). Define $E = \{\{x, y\} : (x, y) \in R_1\}$, then the graph $\Gamma = (X, E)$ is a distance regular graph.

(2) Let $\Gamma = (X, E)$ be a distance regular graph, let us define $R_x \subset X \times X$ by $R_i = \{(x, y) : d(x, y) = i\}$. Then $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is a P-polynomial association scheme.

Proof. It is clearly from the previous discussion. □

Theorem 6.6

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a symmetric association scheme of class d, let $B_1 = (p_{ij}^k)_{0\leqslant j\leqslant d, 0\leqslant k\leqslant d}$ and let $k = p_{11}^0$, then the following conditions are equivalent.

(1) $\mathfrak{X}$ is a P-polynomial association scheme. (with respect to the ordering $R_0, R_1, \dots, R_d$).

(2) B_1^* is the following tri-diagonal matrix

$$\begin{pmatrix}
a_0 & c_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
b_0 & a_1 & c_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & b_1 & a_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & b_2 & \ddots & c_i & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & a_i & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & b_i & \ddots & c_{d-1} & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{d-1} & c_d \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & b_{d-1} & a_d
\end{pmatrix}$$

where $b_i^* \neq 0$ $(0 \leqslant i \leqslant d-1)$, $c_i^* \neq 0$ $(1 \leqslant i \leqslant d)$, $b_0^* = q_{1,1}^0 = m_1$ and $c_1^* = 1$.

(3) Let P be the character table of $\mathfrak{X}$, and let $Q_j = P_i(j)$ for $j \in \{0, 1, \dots, d\}$. Then for each $i \in \{0, \dots, d\}$, there exists a polynomial v_i of degree i, such that $P_i(j) = v_i(Q_j)$, for $j \in \{0, 1, \dots, d\}$.

Before proof this theorem, we give the following remark.

Remark 6.7

All eigenvalues of A_i are determined by the eigenvalues of A_1. Denote the eigenvalues of A_1 by $\theta_0, \theta_1, \dots, \theta_d$.

Proof of Theorem 6.6. (1) $\Rightarrow$ (2) It is clear that $a_0 = p_{10}^0 = 0, b_0 = p_{11}^0 = k$ and $c_1 = p_{10}^0 = 1$. By the assumption, there exists a polynomial $v_i(x)$ of degree i, such that $A_i = v_i(A_1)$. Since $x \cdot v_i(x)$ is a polynomial of degree $i+1$, it is a linear combination of $v_0(x), v_1(x), \dots, v_{i+1}(x)$, so

$$x \cdot v_j(x) = \sum_{\ell=0}^{i+1} c_\ell v_\ell(x).$$

In particular $c_{i+1} \neq 0$ if $i \leqslant d-1$. If we put A_1 to x, then

$$\begin{aligned}\sum_{\ell=0}^{i+1} c_\ell A_\ell &= A_1 v_i(A_1) \\ &= A_1 A_i = \sum_{\ell=0}^{d} p_{1i}^\ell A_\ell,\end{aligned}$$

therefore $p_{1i}^\ell = 0$ if $\ell \geqslant i+2$. We also have $p_{1i}^{i+1} = c_{i+1} \neq 0$. Since $\mathfrak{X}$ is symmetric $k_\ell \cdot p_{1i}^\ell = k_i \cdot p_{1\ell}^i$, so $p_{1\ell}^i = 0$, for every $\ell \geqslant i+2$, and $p_{1,i+1}^i \neq 0$.

(2) $\Rightarrow$ (1) We have

- $a_0 = 0, a_i = p_{1i}^i, \quad (1 \leqslant i \leqslant d);$
- $b_i = p_{1,i+1}^i, \quad (0 \leqslant i \leqslant d-1);$
- $c_i = p_{1,i-1}^i, \quad (1 \leqslant i \leqslant d);$
- $p_{1i}^j = 0$, if $|i-j| \geqslant 2$.

Then

$$A_1 A_i = p_{1i}^{i-1} A_{i-1} + p_{1i}^i A_i + p_{1i}^{i+1} A_{i+1} = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

Define $v_i(x)$ by $v_0(x) = 1, v_1(x) = x$. By

$$x_i v_i(x) = b_{i-1} v_{i-1}(x) + a_i v_i(x) + c_{i+1} v_{i+1}(x),$$

we get $A_{i+1} = v_{i+1}(A_1)$.

(1) $\Leftrightarrow$ (3) Let $E_0, E_1, \dots, E_d$ be the primitive denpotents. And then

$$A_1 = \sum_{j=0}^{d} p_1(j) E_j = \sum_{j=0}^{d} Q_j E_j.$$

For any polynomial

$$v_i(x) = \sum_{\ell=0}^{i} \lambda_\ell x^\ell,$$

we have

$$v_i(A_1) = \sum_{\ell=0}^{i} \lambda_\ell A_1^\ell = \sum_{\ell=0}^{i} (\sum_{j=0}^{d} Q_j^\ell E_j) = \sum_{j=0}^{d} (\sum_{\ell=0}^{i} \lambda_\ell Q_j^\ell) E_j = \sum_{j=0}^{d} v_i(Q_j) E_j.$$

Hence (1) and (3) are equivalent. $\square$

Example 6.8

- Hamming association scheme $H(d,q)$ is a P-polynomial association scheme with

  $c_i = i;$
  $b_i = (d-i)(q-1);$
  $a_i = i(q-2).$

- Johnson association scheme $J(v,d)$ is a P-polynomial association scheme with

  $c_i = i^2;$
  $b_i = (d-i)(v-d-i);$
  $a_i = i(v-2i).$

Exercise 4

Calculate the a_i's, b_i's and c_i's of Hamming scheme and Johnson scheme.

Proposition 6.9

Let $\mathfrak{X}$ be a P-polynomial association scheme, then $\theta_0, \theta_1, \ldots, \theta_d$ of A_1 are all distinct.

Proof. Suppose $\theta_j = \theta_\ell$ with $0 \leqslant j < \ell \leqslant d$. Then by Theorem 6.6 (3), we have

$$P_i(j) = v_i(Q_j) = v_i(Q_e) = P_i(\ell)$$

for every $i \in \{0, 1, \ldots, d\}$. So the j-th row and the l-th row of P are identical. This contradicts the fact that P is non-singular. $(PQ = |X|I)$ □

7 Q-Polynomial Association Scheme

Due to the algebraic structure of Hadamard product, we can introduce a similar construction of P-polynomial association scheme, namely, Q-polynomial association scheme. The concept of Q-polynomial association scheme is due to Delsarte (1973).

Definition 7.1: Q-polynomial association scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a symmetric association scheme. $\mathfrak{X}$ is called a **Q-polynomial association scheme**, if the following conditions are satisfied:

For each $i \geqslant 0$, there exists a polynomial $v_i^*(x)$ of degree i, such that

$$|X|E_i = v_i^*(|X|E_i),$$

where the multiplication is under Hadamard product.

Remark 7.2

There is no good combinatorial interpretation of the concept so far.

Proposition 7.3

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a symmetric association scheme. Then the following are equivalent.

(1) $\mathfrak{X}$ is a Q-polynomial association scheme with respect to some ordering $E_0, E_1, \ldots, E_d$.

(2) B_1^* is the following tri-diagonal matrix

$$\begin{pmatrix} a_0^* & c_1^* & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ b_0^* & a_1^* & c_2^* & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & b_1^* & a_2^* & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & b_2^* & \ddots & c_i^* & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_i^* & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_i^* & \ddots & c_{d-1}^* & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{d-1}^* & c_d^* \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_{d-1}^* & a_d^* \end{pmatrix}$$

where $b_i^* \neq 0$ $(0 \leqslant i \leqslant d-1)$, $c_i^* \neq 0$ $(1 \leqslant i \leqslant d)$, $b_0^* = q_{1,1}^0 = m_1$ and $c_1^* = 1$.

(3) Let $\theta_j^* = Q_1(j)$, $0 \leqslant j \leqslant d$. For each i, there is a polynomial $v_i^*(x)$ of degree i such that $Q_i(j) = v_i^*(\theta_j^*)$ for every $j \in \{0, \ldots, d\}$.

The proof is very similar to the corresponding proposition for P-polynomial association scheme. We left the proof to the reader.

Exercise 5

Prove Proposition 7.3.

Proposition 7.4

Let $\mathfrak{X}$ be a Q-polynomial association scheme. Then the following two assertions hold.

(1) $A_i^* A_1^* = c_{i+1}^* A_{i+1}^* + a_i^* A_i^* + b_{i-1}^* A_{i-1}^*$.

(2) $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are distinct eigenvalues of A_1^*.

Exercise 6

Prove Proposition 7.4.

The polynomials that appear in P-polynomial assocation schemes or in Q-polynomial assocation schemes are important. In fact, they are orthogonal polynomials on a real line with discrete weight. Recall that in usual case, take a, b such that $a < \theta_i < b$ $(0 \leqslant i \leqslant d)$ and define a weight function $w(x)$ by

$$w(x) = \begin{cases} m_1 & \text{if } x = \theta_i, 0 \leqslant i \leqslant d; \\ 0 & \text{otherwise.} \end{cases}$$

For real-valued functions $f_i(x)$ and $f_j(x)$ on $[a, b]$, let

$$(f, g) = \int_a^b f_i(x) f_j(x) w(x)\, dx = \delta_{ij} \alpha_i > 0,$$

where δ is the Kronecker function.

8 Computing the Character Table of Association Scheme

8.1 Association scheme of finite abelian group

Throughout the subsection, let G be a finite Abelian group, and $\mathfrak{X}(G)$ be the group association scheme of G. Since G is abelian, each conjugacy class of G consist one element. We write G additively, and 0 is the identity element of G.

For each $g \in G$, define

$$R_g = \{(x, y) \in G \times G : x - y = g\},$$
$$A_g(x, y) = \delta_{x-y,g} = \begin{cases} 1 & \text{if } x - y = g; \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$\begin{aligned}(A_g, A_h)(x, y) &= \sum_{z\in G} A_g(x, z)A_h(z, y) = \sum_{z\in G} \delta_{x-z,g}\delta_{z-y,h} \\ &= \begin{cases} 1 & \text{if } x - y = g + h; \\ 0 & \text{if } x - y \neq g + h. \end{cases}\end{aligned}$$

So $A_gA_h = A_hA_g = A_{g+h}$.

Let P be the first eigenmatrix of $\mathfrak{X}(G)$ and let E_x $(x \in G)$ be the primitive idempotents of $\mathfrak{X}(G)$. Then

$$A_x = \sum_{g\in G} P_x(g)E_g,$$

and

$$\begin{aligned}\sum_{g\in G} P_{x+y}(g)E_g &= A_{x+y} = A_xA_y \\ &= \left(\sum_{g\in G} P_x(g)E_g\right)\left(\sum_{h\in G} P_y(h)E_h\right) \\ &= \sum_{g\in G} P_x(g)P_y(g)E_g.\end{aligned}$$

Therefore, for each g, x, $y \in G$, we get $P_{x+y}(g) = P_x(g)P_y(g)$.

We define $\Psi_g(x) = P_x(g)$. Then $\Psi_g \in \hat{G}$, where $\hat{G}$ is the irreducible representation characters. In other words, for all x, $y \in G$, we have $\Psi_g(x + y) = \Psi_g(x)\Psi_g(y)$. So $\{\Psi_g : g \in G\} = \hat{G}$. Therefore, P is the same as the character table of G.

We can choose the ordering of the elements of G. So that P is a symmetric matrix. In fact, G is a direct product of cyclic groups (By the structure theorem of finite abelian group). So we can just proof it for $\mathbb{Z}/m\mathbb{Z}$ cases. Let $G = \mathbb{Z}/m\mathbb{Z}$, denote ζ as a primitive with root of 1. Then

$$\Psi_y(x) = P_2(y) = \zeta^{xy}.$$

Then Ψ_g gives a linear character of G. Obviously, $P_x(y) = P_y(x)$. Then we can come to the conclusion that if G is an arbitrary abelian group, P is a symmetric matrix.

In the general case of commutative association scheme, if P is symmetric, then $P = \overline{Q}$. In fact,

$$\frac{P_i(j)}{k_j} = \frac{\overline{Q_j(i)}}{m_j},$$

in the case of commutative association scheme, $k_i = m_j = 1$. For an abelian group G, $\mathfrak{X}(G)$ is self-dual.

8.2 The character table of Hamming association scheme

In this section we will discuss the character table of Hamming association scheme $H(d,q)$, where $H(d,q)$ is a P-polynomial association scheme. We claim that for Hamming association scheme, $H(d,q)$ is self-dual and Q-polynomial as well.

We assume F has a structure of an abelian group. Let X be the d-times copy of F, namely

$$X = \underbrace{F \times \cdots \times F}_{d \text{ times}},$$

and we write $\boldsymbol{x} = (x_1, \dots, x_d) \in X$, $\boldsymbol{y} = (y_1, \dots, y_d) \in X$. Define

$$\omega(\boldsymbol{x}) = |\{i : 1 \leqslant i \leqslant d,\ x_i \neq 0\}|,$$

then the Hamming distance can be defined as

$$d(\boldsymbol{x}, \boldsymbol{y}) = \omega(\boldsymbol{x} - \boldsymbol{y}) = |\{i : (\boldsymbol{x} - \boldsymbol{y})_i \neq 0, \quad i = 1, \dots, d\}.$$

So $R_i = \{(\boldsymbol{x}, \boldsymbol{y}) : \omega(\boldsymbol{x} - \boldsymbol{y}) = i\}$. Then we can define $X_i = \{\boldsymbol{x} \in X : \omega(\boldsymbol{x}) = i\}$, where $X_1, \dots, X_d$ is a partition of X.

Denote F_q as an abelian group of order q. Let $\hat{F}_q$ be the character group of F_q

$$\hat{F}_q = \{\Psi_x : x \in F_q\}$$

with $\Psi_x(y) = \Psi_y(x)$. $X = (F_q)^d$ is also a finite abelian group. For $\boldsymbol{x} = (x_1, \dots, x_d) \in F_q^d$ and $\boldsymbol{y} = (y_1, \dots, y_d) \in F_q^d$,

$$\Psi_{\boldsymbol{x}}(\boldsymbol{y}) = \prod_{i=1}^{d} \Psi_{x_i}(y_i)$$

are linear characters of X. Moreover, $\hat{X} = \{\Psi_{\boldsymbol{x}} : \boldsymbol{x} = (x_1, \dots, x_d) \in F_q^d\}$.

Proposition 8.1

Put $D = \{0, 1, \dots, d\}$. For $j, k \in D$, and $\boldsymbol{u} \in X$, we have

$$K_k(j) := \sum_{\boldsymbol{x} \in X_k} \Psi_{\boldsymbol{u}}(\boldsymbol{x}) = \sum_{i=0}^{k} (-1)^i (q-1)^{k-i} \binom{d-j}{k-1} \binom{j}{i}.$$

Remark 8.2

$K_k(\boldsymbol{u})$ is a polynomial of degree k in $\boldsymbol{u}$. By the generating function of $K_k(\boldsymbol{u})$

$$\begin{aligned}\sum_{k=0}^{\infty} K_k(\boldsymbol{u}) z^k &= (1 + (q-1)z)^{d-u} (1-z)^u \\ &= (1 + (q-1)z)^d \left(\frac{1-z}{1 + (q-1)z} \right)^u,\end{aligned}$$

we have

$$\begin{aligned}K_k(\boldsymbol{u}) &= \sum_{i=0}^{k} (-1)^i (q-1)^{k-i} \binom{d-u}{k-i} \binom{u}{i} \\ &= \sum_{i=0}^{k} (-q)^i (q-1)^{k-i} \binom{d-i}{k-i} \binom{u}{i}.\end{aligned}$$

Here $K_k(\boldsymbol{u})$ is called the Krawtchouk polynomial.

Proof of Proposition 8.1. Write $D^* = D\backslash\{0\} = \{1, 2, \ldots, d\}$. $X_M \subset X_k$, where

$$X_M = \{\boldsymbol{x} \in X_k : x_v \neq 0, \quad v \in M\}$$

Let $M_1, M_2 \subset D^*$, $|M_1| = |M_2| = k$ and $M_1 \neq M_2$. This implies $X_{M_1} \cap X_{M_2} = \varnothing$. We have a partition of X_k

$$X_k = \bigcup_{\substack{M \subset D^* \\ |M|=k}} X_M,$$

where $\boldsymbol{x} = (x_1, \cdots, x_d) \in X_M$. For $v \notin M$, we have $x_v = 0$, and for $v \in M$, we have $x_v \neq 0$. So we have

$$\sum_{\boldsymbol{x} \in X_M} \Psi_{\boldsymbol{u}}(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in X_M} \left(\prod_{v \in X_M} \Psi_{u_v}(x_v) \right) = \prod_{v \in M} \sum_{x_v \in F_q^*} \Psi_{u_v}(x_v),$$

where $\boldsymbol{u} \in X_j$, $j, k \in D$.

Since Ψ_{u_ν} is a linear character of F_q. We get

$$\sum_{x_\nu \in F_q} \Psi_{u_\nu}(x_\nu) = \delta_{u_\nu,0}\, q,$$

where δ is the Kronecker function. So

$$\sum_{x_\nu \in F_q^*} \Psi_{u_\nu}(x_\nu) = \begin{cases} q-1 & \text{if } u_\nu \neq 0; \\ -1 & \text{if } u_\nu = 0. \end{cases}$$

Let $i = |\{\nu \in M : u_\nu \neq 0\}|$. Then we have

$$\prod_{\nu \in M} \sum_{x_\nu \in F_q^*} \Psi_{u_\nu}(x_\nu) = (-1)^i (q-1)^{k-i}.$$

Since $\boldsymbol{u} \in X_j$, for each i, $0 \leqslant i \leqslant d$, there are $\binom{j}{i}\binom{d-j}{k-1}$ number of $M \subset D^*$, such that $i = |\{\nu \in M : u_\nu \neq 0\}|$ and $|M| = k$. So we get

$$\sum_{\boldsymbol{x} \in X_k} \Psi_{\boldsymbol{u}}(\boldsymbol{x}) = \sum_{\substack{M \subset D^* \\ |M|=k}} \sum_{\boldsymbol{x} \in M} \Psi_{\boldsymbol{u}}(\boldsymbol{x}) = \sum_{i=0}^{d} \binom{j}{i} \binom{d-j}{k-i} (-1)^i (q-1)^{k-1}.$$

This completes the proof. □

8.3 The character table of Johnson association scheme

We will give a quick introduction of Johnson association scheme. We just list the contents of several results. The proof of theorems and propositions will be omitted.

Throughout the section, let X be a pair $X = (V, d)$, where V is a set contains v elements, $0 < d \leqslant v/2$. Define $R_i = \{(x, y) \in X \times X : |x \cap y| = d - i\}$. Denote $D = \{0, 1, \ldots, d\}$.

Proposition 8.3

$$k_i = \binom{d}{i} \binom{v-d}{i}.$$

Define $C_i \in \mathfrak{N}(i \in D)$ as

$$C_i = \sum_{\ell=i}^{d} \binom{\ell}{i} A_{d-\ell}. \tag{8}$$

Proposition 8.4

(1) $C_0, C_1, \ldots, C_d$ is a basis of $\mathfrak{N}$.

(2) $(x,y) \in R_{d-k}$ implies $C_i(x,y)$ is the cardinality of the v-elements subset in $x \cap y$;

(3)
$$\binom{d-j}{r-i}\binom{j}{i} = \sum_{\ell=i}^{r}(-1)^{\ell-i}\binom{\ell}{i}\binom{d-\ell}{r-\ell}\binom{j}{\ell};$$

(4)
$$C_rC_s = \sum_{\ell=0}^{\min\{r,s\}}\binom{d-\ell}{r-\ell}\binom{d-\ell}{s-\ell}\binom{v-r-s}{v-d-\ell}C_\ell;$$

(5) For $r = 0, 1, \ldots, d$,
$$C_r = \sum_{j=0}^{r}\binom{d-i}{r-i}\binom{v-r-1}{d-r}E_i;$$

(6) $m_i = \operatorname{rank}(E_i) = \operatorname{rank}(C_i) - \operatorname{rank}(C_{i-1})$.

Theorem 8.5: Kantor [Kan72]

$$\operatorname{rank}(M_i) = |V^{(i)}| = \binom{v}{i}.$$

In [Got66], Gottlieb gives a certain class of incidence matrices. Set $X = V^{(d)}$. For $(x,\xi) \in X \times V^{(i)}$, define
$$M_i(x,\xi) = \begin{cases} 1 & \xi \subset x; \\ 0 & \xi \not\subset x. \end{cases}$$

Proposition 8.6

$C_i = M_i\,{}^tM_i$, where $0 \leqslant i \leqslant d$.

Proof. Since $(x,y) \in R_{d-\ell}$ implies $C_i(x,y) = \binom{\ell}{i}$. We have $x \cap y| = \ell$ and
$$\sum_{\xi \in V^{(i)}} M_i(x,\xi)M_i(y,\xi) = |\{\xi \in V^{(i)} : \xi \subset x \cap y\}| = \binom{\ell}{i}.$$

□

Proposition 8.7

$$m_i = \operatorname{rank}(E_i) = \binom{v}{i} - \binom{v}{i-1},$$

where $0 \leqslant i \leqslant d$.

Theorem 8.8

For Johnson association scheme $J(v, d)$, we have

$$P_j(i) = \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{d-\ell}{d-j} \binom{d-i}{\ell} \binom{v-d+\ell-i}{\ell}$$

and

$$Q_j(i) = m_j k_i^{-1} P_i(j).$$

9 Spherical Embedding of an Association Scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a symmetric association scheme. Let P, Q, E_i $(i = 1, 2, \cdots, d)$ be all real matrices. Write $|X| = n$, and $V = \mathbb{R}^{|X|} = \mathbb{R}^n$. Use $\langle\ ,\ \rangle$ to denote the usual (Euclidean) inner product on V. We write $E = E_1$ and $\theta_i^* = Q_1(i)$.

Definition 9.1: Spherical embedding of $\mathfrak{X}$

The correspondence

$$\begin{aligned} X &\longrightarrow V \\ x &\longmapsto \phi(x) = \sqrt{n} E x \end{aligned}$$

is called the **spherical embedding** of $\mathfrak{X}$. The matrix P is called the **spherical representation** of $\mathfrak{X}$.

We have the following Lemma.

Lemma 9.2

(1) For $(x, y) \in R_i$, we have $\langle \rho(x), \rho(y) \rangle = \theta_i^*$. Here $\theta_i^* = Q_1(i)$.

(2) There exists an $\alpha \in \mathbb{R}$, such that

$$\sum_{y \in \Gamma_i(x)} \rho(y) = \alpha \rho(x),$$

where $\Gamma_i(x) = \{y \in X : (x, y) \in R_i\}$.

Proof. (1) Fix i, $(x, y) \in R_i$. By definition, we have

$$\begin{aligned} \langle \rho(x), \rho(y) \rangle = n \langle Ex, Ey \rangle &= n\, {}^t x E_1 E_1 y = n\, {}^t x E_1 y \\ &= {}^t x \left(\sum_{i=1}^{d} Q_1(i) A_i \right) y = Q_1(i). \end{aligned}$$

(2) By definition, we have

$$\begin{aligned} \sum_{y \in \Gamma_i(x)} \rho(y) = \sqrt{n} E_1 \sum_{y \in \Gamma_i(x)} y &= \sqrt{n} E_1 A_i x \\ &= \sqrt{n} P_i(1) E_1 x = P_i(1) \rho(x). \end{aligned}$$

So $\alpha = P_i(1)$.

□

Note that the set $\{\rho(x) : x \in X\}$ is a subset of $E_1 V = V_1 \subset \mathbb{R}^{|X|}$, where V_1 is a subspace of V of dimension m_1. Since $\langle \rho(x), \rho(x) \rangle = \theta_0^* = Q_1(0) = m_1$, $\{\rho(x) : x \in X\}$ is the sphere of radius $\sqrt{m_1}$ in V_1, this is why we call ρ a spherical representation.

Definition 9.3

The spherical representation $\rho = \rho_E$ is called **non-degenerate**, if $\theta_i^* = \theta_0^*(= m_1)$, $(i = 1, 2, \ldots, d)$.

We have already shown that $|\theta_1^*| = \theta_0^*$. By Definition 9.3, we can see that ρ is non-degenerate if and only if ρ is injective.

Remark 9.4

If $\mathfrak{X}$ is a Q-polynomial association scheme, then $\rho = \rho_E$ is non-degenerate. Because we have shown that $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$ are distinct for Q-polynomial association scheme.

In what follows, we assume that $\rho = \rho_E$ is non-degenerate.

Definition 9.5

Let $x, y \in X$ and $i, j \in \{0, 1, \ldots, d\}$ be arbitrarily fixed. Then we say $\rho = \rho_E$ is **balanced**, if there exists an $\alpha \in \mathbb{R}$, such that

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} \rho(z) - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} \rho(z) = \alpha(\rho(x) - \rho(y)). \tag{9}$$

For $(x, y) \in R_k$, we have $\alpha = \gamma_{ij}^k$, where

$$\gamma_{ij}^k := p_{ij}^k \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_k^*}.$$

If Eq. (9) holds for fixed i and j, then ρ is called $\boldsymbol{\{i, j\}}$**-balanced**.

Proof. Calculate the inner product of $\rho(x)$ and the left hand side of Eq. (9). Then we have

$$p_{ij}^k \theta_i^* - p_{jk}^k \theta_j^* = p_{ij}^k (\theta_i^* - \theta_j^*).$$

Calculate the inner product of $\rho(x)$ and the right hand side of Eq. (9). Then we have

$$\alpha(\theta_0^* - \theta_k^*).$$

□

Proposition 9.6

Let Δ_E be the representation graph for $E = E_1$. Then the following two conditions are equivalent.

(1) $\rho = \rho_E$ is non-degenerate;

(2) $\Delta = \Delta_E$ is a connected graph.

Proof. We have already shown that Δ is connected if and only if the dimension of eigenspace for eigenvalue $Q_1(0)$ is 1 in B_1^*, if and only if $Q_1(j) \neq Q_1(0)$ for all $j \neq 0$. □

Now we show another proof of (1) ⇒ (2) in Proposition 9.6.

Another proof of (1) ⇒ (2) in Proposition 9.6. Let Δ' be a connected component of $\Delta, E_{\Delta'} = \sum_{i \in \Delta'} E_i$, $A^* = A_1^*$, $x_0 \in X$ is fixed, and $\mathfrak{N}^* = \mathfrak{N}^*(x_0)$. First we show

$$E_i A^* = E_{\Delta'} A^* E_{\Delta'} = A^* E_{\Delta'}. \tag{10}$$

Note that $E_iA^*E_j = 0$ if and only if $q^j_{1i} \neq 0$. Then

$$E_{\Delta'}A^*I = \left(\sum_{i\in\Delta'} E^i\right) A^* \left(\sum_{j=0}^{d} E_j\right) = \sum_{\substack{i\in\Delta' \\ 0\leqslant j\leqslant d}} E_iA^*E_j$$
$$= \sum_{i,j\in\Delta'} E_iA^*E_j = E_{\Delta'}A^*E_{\Delta'}.$$

Similarly, we have $IA^*E_{\Delta'} = E_{\Delta'}A^*E_{\Delta'}$. So Eq. (10) holds.

Set $E_{\Delta'} = \sum_{i=0}^{d} \alpha_i A_i$. Then

$$0 = E_{\Delta'}A^* - A^*E_{\Delta} = \sum_{i=1}^{d} \alpha_i(A_iA^* - A^*A_i).$$

while

$$(A_iA^* - A^*A_i)(x, y) = A_i(x_0, y)A^*(y, y) - A^*(x_0, x_0)A_i(x_0, y)$$
$$= \begin{cases} \theta^*_i - \theta^*_0, & \text{if } (x, y) \in R_i, \\ 0, & \text{if } (x, y) \notin R_i. \end{cases}$$

Since ρ is non-degenerate, $\theta^*_i \neq \theta^*_0$. For any $i \neq 0$, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 0$. So $E_{\Delta'} = \alpha_0 I$. Since $E_{\Delta'}$ is an idempotent, then $\alpha_0^2 = \alpha_0$. So $\alpha_0 = 0$ or 1. We notice that $\alpha_0 = 0$ implies $\Delta' = \varnothing$, and $\alpha_0 = 1$ implies $\Delta' = \{0, 1, \ldots, d\}$. Therefore Δ is connected. □

Theorem 9.7

Let $\mathfrak{X}$ be a symmetric association scheme, E be a primitive idempotent, $\rho = \rho_E$ and $\Delta = \Delta_E$. The following are equivalent.

(1) $\rho = \rho_E$ is balanced;

(2) $\Delta = \Delta_E$ is a tree.

We see a corollary first, which can be verified immediately from Theorem 9.7.

Corollary 9.8

Let $\mathfrak{X}$ be a Q-polynomial association scheme with respect to $(E_1, \ldots, E_d)$. Then $\rho = \rho_E$ is balanced.

Proof. The assumption of $\mathfrak{X}$ is a Q-polynomial association scheme implies $\Delta_E = \Delta_{E_1}$ is a path, which implies Δ_E is a tree by Theorem 9.7. Then $\rho = \rho_E$ is balanced. □

Proof of Theorem 9.7. Recall that A^* is the diagonal matrix.

$$A^*(x, x) = \begin{cases} Q_1(i), & \text{if } (x_0, x) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L} = \mathcal{L}(x_0) = \operatorname{span}\{MA^*N - NA^*M : M, N \in \mathfrak{N}\}$. We need the following lemma.

Lemma 9.9

Let $\mathfrak{N}^* = \mathfrak{N}^*(x_0)$ be the dual Bose-Mesner algebra, $A^* = A^*_1 \in \mathfrak{N}^*$.

(1) $\{E_iA^*E_j - E_jA^*E_i : 0 \leqslant i, j \leqslant d,\ \ i \neq j,\ \ q^j_{1i} \neq 0\}$ is a basis of $\mathcal{L}$.

(2) The subset $\{A^*A_k - A_kA^* : 1 \leqslant k \leqslant d\}$ of $\mathcal{L}$ is linearly independent. Moreover, $E_iA^*E_j = 0$ if and only if $q^j_{1i} \neq 0$.

Proof of Lemma 9.9. (1) Note that $M = \sum_{i=0}^{d} \alpha_i E_i$ and $N = \sum_{j=0}^{d} \beta_j E_j$. By Corollary 4.6 (2)

$$E_i A^* E_j = 0 \Leftrightarrow q_{1i}^j = 0,$$

so

$$MA^*N - NA^*M = \sum_{\substack{i,j \\ q_{1i}^j \neq 0}} \alpha_i \beta_j (E_i A^* E_j - E_j A^* E_i).$$

So

$$\mathcal{L} = \operatorname{span}\{E_i A^* E_j - E_j A^* E_i : 0 \leqslant i, j \leqslant d, \quad i \neq j, \quad q_{1i}^j \neq 0\}.$$

Suppose

$$\sum_{\substack{i,j,i\neq j \\ q_{1i}^j \neq 0}} \alpha_{ij} (E_i A^* E_j - E_j A^* E_i) = 0.$$

Multiply E_i from the left and multiply E_j from the right, then $q_{1i}^j \neq 0 \Leftrightarrow E_i A^* E_j \neq 0$, we have $\alpha_{ij} = 0$.

(2) As we have shown in another proof of Proposition 9.6, we have

$$(A^*A_k - A_k A^*)(x, y) = \begin{cases} \theta_0^* - \theta_k^*, & \text{if } (x_0, y) \in R_k, \\ 0, & \text{if } (x_0, y) \notin R_k. \end{cases}$$

Suppose $\sum_{k=1}^{d} \alpha_k (A^*A_k - A_k A^*) = 0$, then calculate (x_0, y)-entry (with $(x_0, y) \in R_k$), we obtain $\alpha_k(\theta_0^k - \theta_k^*) = 0$. Since ρ is non-degenerate and $\theta_0^* \neq \theta_k^*$, so $\alpha_k = 0$.

□

Corollary 9.10

(1) $\dim \mathcal{L}$ is just the number of edges in the graph $\Delta = \Delta_E$.

(2) $d \leqslant \dim \mathcal{L}$ and $d = \dim \mathcal{L}$ if and only if Δ_E is a tree.

Proof of Corollary 9.10. The proof of (1) is from Lemma 9.9.

Lemma 9.9 (2) shows that $d \leqslant \dim \mathcal{L}$. Since ρ is non-degenerate, Δ is connected. So $\dim \mathcal{L} = d$ if and only if Δ is a tree. Hence (2). □

Now we return to the proof of Theorem 9.7. By the discussion above, we know that Δ is a tree if and only if

$$d = \dim \mathcal{L},$$

if and only if

$$\mathcal{L} = \operatorname{span}\{A^*A_k - A_k A^* : 1 \leqslant k \leqslant d\},$$

if and only if for all $i, j \in \{0, 1 \cdots, d\}$, there exist γ_{ij}^k, such that

$$A_i A^* A_j - A_j A^* A_i = \sum_{k=1}^{d} \gamma_{ij}^k (A^* A_k - A_k A^*). \tag{11}$$

Suppose Eq. (11) holds, then

$$\gamma_{ij}^k = p_{ij}^k \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_k^*}$$

since $(x_0, y) \in R_k$ implies for the (x_0, y)-entry of Eq. (11), the right hand side is

$$\gamma_{ij}^k (\theta_0^* - \theta_k^*),$$

and the left hand side is

$$\sum_{z \in X} A_i(x_0, z) A^*(z, z) A_j(z, y) - \sum_{z \in X} A_j(x_0, z) A^*(z, z) A_i(z, y) = p_{ij}^k (\theta_i^* - \theta_j^*).$$

Hence we prove the first part of Theorem 9.7.Before we prove the second part, we need another lemma.

Lemma 9.11

Let $\rho = \rho_E$, $A^* = A_1^*(x_0) \in \mathfrak{N}^*(x_0)$. Then

(1) $A^*(z,z) = \langle \rho(x_0), \rho(z) \rangle$.

(2) $(A^*A_k - A_kA^*)(x,y) = \begin{cases} \langle \rho(x_0), \rho(x) - \rho(y) \rangle, & \text{if } (x,y) \in R_k, \\ 0, & \text{if } (x,y) \notin R_k. \end{cases}$

Proof. (1) Since $(x_0, z) \in R_k$ implies $A^*(z,z) = Q_1(k) = \theta_k^*$, we obtain $A^*(z,z) - \langle \rho(x_0), \rho(z) \rangle$ by Lemma 9.2.

(2) As we have shown in another proof of Proposition 9.6, we have

$$\begin{aligned} (A^*A_k - A_kA^*)(x,y) &= A^*(x,x)A_k(x,y) - A_k(x,y)A^*(y,y) \\ &= \begin{cases} A^*(x,x) - A^*(y,y) & \text{if } (x,y) \in R_k; \\ 0, & \text{if } (x,y) \notin R_k. \end{cases} \end{aligned}$$

□

Now we return to the proof of the second part of Theorem 9.7.
The left hand side of Eq. (11) is given by

$$\begin{aligned} (A_iA^*A_j - A_jA^*A_i)(x,y) &= \sum_{z \in X} A_i(x,z)A^*(z,z)A_j(z,y) - \sum_{z \in X} A_j(x,z)A^*(z,z)A_i(z,y) \\ &= \sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} A^*(z,z) - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} A^*(z,z) \\ &= \left\langle \rho(x_0), \sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} \rho(z) - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} \rho(z) \right\rangle. \end{aligned}$$

For $(x,y) \in R_k$, the (x,y)-entry of right hand side of Eq. (11) is $\gamma_{ij}^k \langle \rho(x_0), \rho(x) - \rho(y) \rangle$. So Eq. (11) holds if and only if Δ is a tree, if and only if the balanced condition

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} \rho(z) - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} \rho(z) = \alpha(\rho(x) - \rho(y))$$

holds.

Theorem 9.12

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a P-polynomial association scheme. Let $\rho = \rho_E$ be the spherical representation of $\mathfrak{X}$ with respect to $E = E_1$. Let ρ be non-degenerate and $\{1,2\}$-balance. Then $\mathfrak{X}$ is a Q-polynomial association scheme with respect to E_1.

Proof. This proof can be divided into four steps (I) to (IV).

(I) Let $(x,y) \in R_k$, then the balanced condition holds.

(II) For $A^* = A_1^*(x_0)$, then

$$A_1A^*A_2 - A_2A^*A_1 = \sum_{k=1}^{d} \gamma_{1,2}^k (A^*A_k - A_kA^*) \tag{12}$$

(III) Let $A = A_1$, there exists $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A) = 0 \tag{13}$$

(IV) Δ is a path (i.e., $\mathfrak{X}$ is a Q-polynomial association scheme with respect to E).

Proof of Step (I).

$$\sum_{z\in\Gamma_1(x)\bigcap\Gamma_2(y)}\rho(z)-\sum_{z\in\Gamma_2(x)\bigcap\Gamma_1(y)}\rho(z)=\alpha(\rho(x)-\rho(y)),$$

with $\alpha=\gamma_{1,2}^k=p_{1,2}^k\frac{\theta_1^*-\theta_2^*}{\theta_0^*-\theta_k^*}$. □

Proof of Step (II). Let $(x,y)\in R_k$. Consider the entries of both sides of Eq. (12). From left side, we have

$$\begin{aligned}&(A_1A^*A_2-A_2A^*A_1)(x,y)\\&=\sum_{z\in X}A_1(x,z)A_1^*(z,z)A_2(z,y)-\sum_{z\in X}A_2(x,z)A_1^*(z,z)A_1(z,y)\\&=\sum_{z\in\Gamma_1(x)\bigcap\Gamma_2(y)}\langle\rho(x_0),\rho(z)\rangle-\sum_{z\in\Gamma_2(x)\bigcap\Gamma_1(y)}\langle\rho(x_0),\rho(z)\rangle\\&=\langle\rho(x_0),\alpha(\rho(x)-\rho(y))\rangle\end{aligned}$$

while,

$$\begin{aligned}&\sum_{l=1}^{d}\gamma_{1,2}^l(A^*A_l-A_lA^*)(x,y)\\=&\gamma_{1,2}^k(A^*(x,x)-A^*(y,y))\\=&\gamma_{1,2}^k\langle\rho(x_0),\rho(x)-\rho(y)\rangle.\end{aligned}$$

Hence Step (II) holds.

□

Proof of Step (III). Since $\mathfrak{X}$ is a P-polynomial association scheme $p_{1,2}^k=0$ for $k\in\{4,5,\dots,d\}$, so $\gamma_{1,2}^k=0$. Then, by Eq. (12) we have

$$\begin{aligned}&A_1A^*A_2-A_2A^*A_1\\=&\gamma_{1,2}^3(A^*A_3-A_3A^*)+\gamma_{1,2}^2(A^*A_2-A_2A^*)+\gamma_{1,2}^1(A^*A_1-A_1A^*).\end{aligned}$$

In the case that $\gamma_{1,2}^3\neq 0$. Since $\mathfrak{X}$ is P-polynomial association scheme, there exists a polynomial $v_i(x)$satisfied that

$$A_i=v_i(A_1)=v_i(A).$$

Suppose that

$$v_3(x)=a_3x^3+a_2x^2+a_1x+a_0$$

with $a_3\neq 0$. Then

$$\begin{aligned}A^*A_3-A_3A^*&=A^*v_3(A)-v_3(A)A^*\\&=A^*(a_3A^3+a_2A^2+a_1A+a_0I)-(a_3A^3+a_2A^2+a_1A+a_0I)A^*\\&=a_3(A^*A^3-A^3A^*)+a_2(A^*A^2-A^2A^*)+a_1(A^*A-AA^*)\end{aligned}\tag{14}$$

So by Eqs. (13) and (14), $A^3A^*-A^*A^3$ is a linear combination of $A^2A^*A-AA^*A^2$, $A^2A^*-A^*A^2$, AA^*-A^*A.

In the case that $\gamma_{1,2}^3=0$. We obtain that $\gamma_{1,2}^3=0$, $\rho_{1,2}^3\neq 0$ since $\mathfrak{X}$ is P-polynomial association scheme. It implies that $\theta_1^*=\theta_2^*$, then $\gamma_{1,2}^k=0$ for $k=1,2,\dots,d$, by Step (I). And by Step (II), $A_1A^*A_2-A_2A^*A_1=0$.

Mulitiply E_i from the left and E_j from the right, we have

$$\begin{aligned}0&=E_i(A_1A^*A_2-A_2A^*A_1)E_j\\&=(P_1(i)P_2(j)-P_2(i)P_1(j))\,E_iA^*E_j.\end{aligned}$$

Let $\theta_k = P_1(k)$, then $P_2(k) = v_2(\theta_k)$, so $(\theta_i v_2(\theta_j) - v_2(\theta_i)\theta_j)E_iA^*E_j = 0$. Consider $i, j (i \neq j)$ such that $q_{1,i}^j \neq 0$, this implies $E_iA^*E_j \neq 0$. Since $v_2(x) = \frac{1}{c_2}(x^2 - a_1x - k)$, we get $\theta_i v_2(\theta_j) - v_2(\theta_i)\theta_j = \frac{1}{c_2}(\theta_j - \theta_i)(\theta_i\theta_j + k)$. Since $\theta_0, \theta_1, \dots, \theta_d$ are distinct real numbers, for $i, j (i \neq j)$ such that $q_{1,i}^j \neq 0$, we get $\theta_i\theta_j = -k \neq 0$.

Since ρ is non-degenerate and $q_{1,i}^j \neq 0$ and i, j are connected by an edge in Δ, then there exists $j' \in \{0, \dots, d\}$ such that $q_{1,i}^{j'} \neq 0$. So $\theta_i\theta_j = \theta_i\theta_{j'} = -k \neq 0$, it implies $\theta_j = \theta_{j'}$ which is impossible.

□

Proof of Step (IV). Since ρ is non-degenerate and Δ is connected, we want to show that each degree of Δ is at most 2. Choose a vertex i of Δ such that $q_{1,i}^j \neq 0$, then $E_iA^*E_j \neq 0$. By Eq. (13), multiply E_i from the left and E_j from the right, then we have:

$$\begin{aligned} 0 &= E_i(A^3A^* - A^*A^3 - (\beta+1)(A^2A^*A - AA^*A^2) - \gamma(A^2A^* - A^*A^2) - \delta(AA^* - A^*A))E_j \\ &= (\theta_i^3 - \theta_j^3 - (\beta+1)(\theta_i^2\theta_j - \theta_i\theta_j^2) - \gamma(\theta_i^2 - \theta_j^2) - \delta(\theta_i - \theta_j))E_iA^*E_j \\ &= (\theta_i - \theta_j)(\theta_i^2 - \beta\theta_i\theta_j + \theta_j^2 - \gamma(\theta_i + \theta_j) - \delta)E_iA^*E_j \end{aligned}$$

So if we fix θ_i then θ_j is a root of a quadratic equation (they are all distinct), so there are at most 2 verties adjacent to i. □

□

Question 9.1

Under what condition, Q-polynomial condition implies P-polynomial condition?

10 Some Open Problems

10.1 Dual balanced condition

Question 10.1

We know that balanced condition is equivalent to Δ_E is a tree. Does it have a dual balanced condition to show the equivalence of Δ_A is a tree?

10.2 Classification of P-polynomial commutative association scheme with valency $k_1 = 3$

See

- Biggs - Boshier - Shawe - Taylor. [BBST86] Cubic ($k_1 = 3$) distance regular graphs.
- Bannai - Ito. [BI87a, BI88, BI87b, BI89] Distance-regular graph with fixed valency I-IV.

10.3 Bannai-Ito conjucture

Conjecture 10.2: Bannai-Ito Conjucture

For each fixed $k \geqslant 3$, there are only finitely many P-polynomial association scheme.

- Bang-Dubickas-Koolen-Moulton [BKP15] pointed that there are only finitely many distance-regular graphs of fixed valency greater than two.
- Yamazaki [Yam98] completes the case on symmetric association scheme with $k_1 = 3$ (primitive or with some weaker condition but with some conditions).

- Bannai-Bannai [BB06] complete the case on primitive symmetric associations with $m_1 = 3$.

  We want to drop this condition to $m_1 = 4$, or Q-polynomial association scheme cases.

- Martin-Williford [MMW07] show that there are finitely many Q-polynomial association schemes with given multiplicity at least 3 ($m_1 \geqslant 3$).

10.4 Remaining problems

Problem 10.3

- Try to classify Q-polynomial association scheme with $m_1 = 3$.
- Try to classify symmetric association scheme with $m_1 = 3$ with balanced condition. (Question: Are there only finitely many?)
- Try to classify symmetric association scheme with $m_1 = 3$. On the other word, ρ_E is non-degenerate.

van Dam-Koolen-Park completes the case of partially metric association scheme with a multiplicity three. Can we study the problem in this paper by [BB06]'s paper? □

11 Linear programming (after Delsarte)

11.1 Linear programming method

We let $(X, \{R_i\}_{0\leqslant i\leqslant d})$ be a symmetric association $D = \{0, 1, \ldots, d\}$, $D^* = D \setminus \{0\}$, $M \subset D$, $0 \in M$ and $M^* = M \setminus \{0\}$. Let C be a $(d+1) \times (d+1)$ matrix whose rows and columns are parametrized by D, $C = (C_j(i))_{0\leq i,j\leq d}$ with $C_0(i) = 1$. We consider the following two problems.

Problem: (C, M)

Under

Condition 1

$$\sum_{i\in M} a_i C_j(i) \geq 0, \quad j \in D^*;$$
$$a_i \geq 0, \quad i \in M^*;$$

makes

$$g = \sum_{i\in M} a_i C_0(i) = \sum_{i\in M} a_i$$

maximum.

Problem: $(C, M)'$

Under

Condition 2

$$\sum_{j \in D} \alpha_j C_j(i) \le 0, \quad i \in M^*;$$
$$\alpha_j \ge 0, \quad j \in D^*;$$

makes

$$\gamma = \sum_{j \in D} \alpha_j C_j(0)$$

minimum.

Definition 11.1

- $|M| = m + 1$, a vector $\boldsymbol{a} = (a_0, a_1, \ldots, a_m) \in \mathbb{R}^{m+1}$ is indexed by M. If $\boldsymbol{a}$ satisfies $a_0 = 1$ and Condition 1, then $\boldsymbol{a}$ is called a **program** of (C, M).

  If $\boldsymbol{a}$ gives the max value of g, it is called the **max program**.
- $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_d) \in R^{d+1}$ is indexed by D. If $\boldsymbol{\alpha}$ satisfies $\alpha_0 = 1$ and Condition 2, then $\boldsymbol{\alpha}$ is called a **program** of $(C, M)'$.

  If $\boldsymbol{\alpha}$ gives the min value of γ, it is called the **min program**.
- (C, M) is called **feasible** if there is a program $\boldsymbol{a}$ of (C, M).
- $(C, M)'$ is called **feasible** if there is a program $\boldsymbol{\alpha}$ of $(C, M)'$.

Proposition 11.2

Let $\boldsymbol{a}$ and $\boldsymbol{\alpha}$ be programs of (C, M) and $(C, M)'$ respectively, then $g \le \gamma$.

Proof. By Condition 1,

$$a_0 C_j(0) + \sum_{i \in M^*} a_i C_j(i) \ge 0.$$

So,

$$\sum_{i \in M^*} a_i C_j(i) \ge -a_0 C_j(0) = -C_j(0), \quad j \in D^*.$$

So,

$$\sum_{j \in D^*} \alpha_j \Big(\sum_{i \in M^*} a_i C_j(i) \Big) \ge -\sum_{j \in D^*} \alpha_j C_j(0) = 1 - \sum_{j \in D} \alpha_j C_j(0) = 1 - \gamma. \tag{15}$$

While, by Condition 2

$$\sum_{j \in D^*} \alpha_j C_j(i) \le -\alpha_0 C_0(i) = -1, \; i \in M^*.$$

So,

$$\sum_{i \in M^*} a_i \Big(\sum_{j \in D^*} \alpha_j C_j(i) \Big) \le -\sum_{i \in M^*} a_i = 1 - g. \tag{16}$$

By Eqs. (15) and (16), $1 - \gamma \le 1 - g$ and thus $g \le \gamma$. □

Theorem 11.3

Let Problem (C, M) and Problem $(C, M)'$ be feasible. Then there are a maximum program and a minimum program with $\max g = g_0$ and $\min \gamma = \gamma_0$ and $g_0 = \gamma_0$.

Before proving the theorem, we need to do some preparations.

Definition 11.4

$\Omega \in R^n$ is called a **closed convex cone** if the following are satisfied:

(1) For every $x, y \in \Omega$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1-\lambda)y \in \Omega$;

(2) For every $x \in \Omega$ and $\alpha \geq 0$, we have $\alpha x \in \Omega$.

Let Ω be a closed convex cone, define

$$\Omega^* = \{u \in R^n : u \cdot x \geq 0, \text{ for all } x \in \Omega\}.$$

Then Ω^* is also a closed convex cone and it is known that $(\Omega^*)^* = \Omega$.

For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we say $x \geq 0$ if $x_i \geq 0,\ 1 \leq i \leq n$.

Theorem 11.5

Let A be a $m \times n$ matrix, $c \in \mathbb{R}^n$. Suppose $c \cdot u \geq 0$ for each $u \in \mathbb{R}^n$ with $Au \geq 0$, then there exists $x \in \mathbb{R}^m,\ x \geq 0$ such that $c = {}^tAx$.

Lemma 11.6

Let a be the max program of Problem (C, M). Let α be the min program of Problem $(C, M)'$. Then

$$\alpha_j \left(\sum_{i \in M} a_i c_j(i) \right) = 0 \quad \text{for } j \in D^*, \tag{17}$$

and

$$a_i \left(\sum_{j \in M} \alpha_j c_j(i) \right) = 0 \quad \text{for } i \in M^*. \tag{18}$$

Conversely, if Eqs. (17) and (18) hold for a and α, respectively, then a is the max program of (C, M) and α is the min program of $(C, M)'$.

Proof. Let g_0 be the max program of g. Let γ_0 be the min program of γ. So,

$$g_0 = \gamma_0 = \sum_{i \in M} a_i = \sum_{j \in D} \alpha_j C_j(0).$$

Since $a_i \geq 0$ (by Condition 2),

$$\begin{aligned}
0 &\geq \sum_{i\in M^*} a_i \left(\sum_{j\in D} \alpha_j c_j(i)\right) \\
&= \sum_{j\in D}\sum_{i\in M^*} a_i C_j(i) \\
&= \sum_{j\in D} \alpha_j \left(\sum_{i\in M} a_i C_j(i) - a_0 C_j(0)\right) \\
&= \sum_{j\in D^*} \alpha_j \sum_{i\in M} a_i C_j(i) + \sum_{i\in M} \alpha_0 a_i C_0(i) - \sum_{j\in D} \alpha_j a_0 C_j(0) \\
&= \sum_{j\in D^*} \alpha_j \sum_{i\in M} a_i C_j(i) + g_0 - \gamma_0 = \sum_{j\in D^*} \alpha_j \sum_{i\in M} a_i C_j(i) \geq 0.
\end{aligned}$$

So

$$\sum_{i\in M^*} a_i \sum_{j\in D} \alpha_j C_j(i) = \sum_{j\in D^*} \alpha_j \sum_{i\in M} a_i C_j(i) = 0.$$

For $i \in M^*, a_i \geq 0$, we have

$$\sum_{j\in D} \alpha_j C_j(i) \leq 0.$$

For $j \in D^*, \alpha_i \leq 0$, we have

$$\sum_{i\in M} a_i C_j(i) \geq 0.$$

Hence Eqs. (17) and (18) hold.

Conversely, if these two relation hold then a is the max program of (C, M) and α is the min program of $(C, M)'$. □

12 Subsets of an Association Scheme

12.1 Subsets of an association scheme

Definition 12.1

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a symmetric association scheme, $D = \{0, 1, \ldots, d\}$ be the index set, $P$and Q be the first and second eigenmatrices of $\mathfrak{X}$. For a subset $Y \subset X$, define $\boldsymbol{a}_Y = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1}$,

$$a_i = \frac{1}{|\mathrm{Y}|}|R_i \cap (Y \times Y)|, \quad \text{for all } i \in D.$$

Here $\boldsymbol{a}_Y$ is called the **inner distribution** of Y, and $\boldsymbol{a}_Y^* = \boldsymbol{a}_Y Q = (a_0^*, a_1^*, \ldots, a_d^*)$ is called the **dual distribution** of Y. B_Y is a matrix defined by $X \times D$, its entries are defined by

$$B_Y(x, i) = |R_i \cap (\{x\} \times Y)| = |Y \cap \Gamma_i(x)|.$$

B_Y is called the **outer distribution** of Y. Define ψ_Y as the **characteristic vector** of Y,

$$\psi_Y(x) = \begin{cases} 1 & \text{if } x \in Y; \\ 0 & \text{if } x \notin Y. \end{cases}$$

Sometimes we write $\Psi_{\{x\}} = \Psi_x$. For a vector $\boldsymbol{u} = (u_1, \ldots, u_r) \in \mathbb{R}^r$, we define a diagonal matrix,

$$\Delta_{\boldsymbol{u}} := \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_r \end{pmatrix}$$

Proposition 12.2

$Y \subset X$, $\boldsymbol{a}_Y = (a_0, a_1, \ldots, a_d)$, then

(1) $a_0 = 1$;

(2) $a_0^* = \sum_{i=0}^d a_i = |Y|$.

Proof. It is clear by the definition. □

Proposition 12.3

(1) $a_1 = \frac{1}{|Y|} {}^t\psi_Y A_i \psi_Y$;

(2) $B_Y = [A_0\psi_Y, A_1\psi_Y, \ldots, A_d\psi_Y]$;

(3) $\boldsymbol{a}_Y = \frac{1}{|Y|} {}^t\psi_Y B_Y$.

Proof. (1)

$$\frac{1}{|Y|} {}^t\psi_Y A_i \psi_Y = \frac{1}{|Y|} \sum_{(x,y)\in Y\times Y} A_i(x,y) = a_i$$

(2) For $(x, i) \in X \times D$,

$$(A_i\psi_Y)(x) = \sum_{z\in Y} A_i(x,z)\psi_Y(z) = |Y \cap \Gamma_i(x)| = B_Y(x,i)$$

So $A_i\psi_Y$ is the i-th column of B_Y.

(3)

$$\begin{aligned} &\frac{1}{|Y|} {}^t\psi_Y B_Y \\ =&[\frac{1}{|Y|} {}^t\psi_Y A_0\psi_Y, \frac{1}{|Y|} {}^t\psi_Y A_1\psi_Y, \ldots, \frac{1}{|Y|} {}^t\psi_Y A_d\psi_Y] \\ =&(a_0, a_1, \ldots, a_d) = \boldsymbol{a}_Y. \end{aligned}$$

□

Theorem 12.4

$${}^tB_Y B_Y = \frac{|Y|}{|X|} {}^tP \Delta_{\boldsymbol{a}_Y^*} P.$$

Proof. By Proposition 12.3 (2),

$$\begin{aligned} ({}^tB_Y B_Y)(i,j) = {}^t(A_i\psi_Y) \cdot (A_j\psi_Y) &= {}^t\psi_Y A_i A_j \psi_Y \\ &= \sum_{k=0}^d p_{i,j}^k \, {}^t\psi_Y A_k \psi_Y = |Y| \sum_{k=0}^d p_{i,j}^k a_k. \end{aligned}$$

Since $\boldsymbol{a}_Y = \frac{1}{|X|}\boldsymbol{a}_Y^* P$, $a_k = \frac{1}{|X|}\sum_{\ell=0}^d a_\ell^* P_k(\ell)$,

$$({}^t B_Y B_Y)(i,j) = \frac{|Y|}{|X|}\sum_{k=0}^d p_{i,j}^k \sum_{\ell=0}^d a_\ell^* P_k(\ell) = \frac{|Y|}{|X|}\sum_{\ell=0}^d a_\ell^* \sum_{k=0}^d p_{i,j}^k P_k(\ell).$$

So

$$({}^t B_Y B_Y)(i,j) = \frac{|Y|}{|X|}\sum_{\ell=0}^d a_l^* P_i(\ell)P_j(\ell) = \frac{|Y|}{|X|}({}^t P \Delta_{\boldsymbol{a}_Y^*} P)(i,j).$$

□

Corollary 12.5

Rank of $B_Y = |\{i : a_i^* \neq 0\}|$.

Theorem 12.6

Let $Y \subset X$, and then $a_k^* \geq 0$ for every $k \in D$. The following 3 conditions are equivalent.

(1) $a_k^* = 0$;

(2) $B_Y Q_k = \mathbf{0}$;

(3) $E_k \psi_Y = \mathbf{0}$.

Proof. (1) ⇔ (2) By Theorem 12.4,

$$({}^t Q\, {}^t B_Y) B_Y Q = \frac{|Y|}{|X|}\, {}^t Q({}^t P \Delta_{\boldsymbol{a}_Y^*} P) Q = |X| \cdot |Y| \Delta_{\boldsymbol{a}_Y^*}.$$

Compare the diagonal entry of both sides,

$$\sum_{x \in X} ((B_Y Q)(x,k))^2 = |X||Y| a_k^*, \quad k = 0, 1, \ldots, d.$$

Hence $(B_Y Q)(x,k) = 0$, for all $x \in X$ and then $a_k^* = 0$. Moreover $a_k^* = 0$ if and only if $(B_Y Q)(x,k) = 0$, for all $x \in X$.

(1) ⇔ (3) By Proposition 12.3 (1), $a_i = \frac{1}{|Y|}\, {}^t \psi_Y A_i \psi_Y$. Then we have

$$\begin{aligned} a_k^* &= \sum_{i=0} d a_i Q_k(i) = \sum_{i=0} d \frac{1}{|Y|}\, {}^t \psi_Y A_i \psi_Y Q_k i \\ &= \frac{|X|}{|Y|}\, {}^t \psi_Y E_k \psi_Y = \frac{|X|}{|Y|}\, {}^t \psi_Y E_k E_k \psi_Y \\ &= \frac{|X|}{|Y|}\, {}^t (E_k \psi_Y)(E_k \psi_Y). \end{aligned}$$

□

Definition 12.7

Let $\boldsymbol{a}_Y = (a_0, a_1, \ldots, a_d)$ be the inner distribution of Y and $\boldsymbol{a}^*_Y = (a^*_0, a_1, \ldots, a^*_d)$ be the dual inner distribution of Y. Then,

(1) $\delta = \min\{i : a_i \neq 0, i \geq 1\}$, $\delta^* = \min\{i : a*_i \neq 0, i \geq 1\}$ are called **minimum distance** and **dual minimum distance** of Y respectively;

(2) $s = |\{i : a_i \neq 0, i \geq 1\}|$, $s^* = \{i : a*_i \neq 0, i \geq 1\}$ are defined as the **degree** and **dual degree** of Y;

(3) $t = \max\{i : a^*_1 = a^*_2 = \cdots = a^*_i = 0, i \geq 1\} = \delta^* - 1$ is called **strength** of Y.

We introduce some notations. For any $\boldsymbol{u} = (u_0, u_1, \ldots, u_d) \in \mathbb{R}^{d+1}$, we can also define $t(\boldsymbol{u}) = \max\{i : u_1 = u_2 = \cdots = u_i = 0\}$ and $s(\boldsymbol{u}) = |\{i : u_i \neq 0, 1 \leq i \leq d\}|$.

Theorem 12.8: Mac Williams inequality

$\boldsymbol{u} = (u_0, u_1, \ldots, u_d) \in \mathbb{R}^{d+1}, u_0 \neq 0$

(1) Let $\mathfrak{X}$ be a P-polynomial association scheme then $s(\boldsymbol{u} \cdot Q) \geq [\frac{t(\boldsymbol{u})}{2}]$;

(2) Let $\mathfrak{X}$ be a Q-polynomial association scheme then $s(\boldsymbol{u} \cdot P) \geq [\frac{t(\boldsymbol{u})}{2}]$.

Remark 12.9

Proof of (2) is very similar to that of (1). So, we just discuss proof of (1).

Corollary 12.10

Let $\mathfrak{X}$ be a Q-polynomial association scheme. If $Y \subset X$, then $s \geq \left[\frac{t}{2}\right]$.
ceiling?

Proof. Set $\boldsymbol{u} = \boldsymbol{a}^*_Y = \boldsymbol{a}_Y Q$, $u_0 = a^*_0 > 0$. So $s = s(\boldsymbol{a}_Y) = s(\boldsymbol{u}P) \geq \left[\frac{t(\boldsymbol{u})}{2}\right] = \left[\frac{t(\boldsymbol{a}^*_Y)}{2}\right] = \left[\frac{t}{2}\right]$. □

Now we turn to the proof of Mac Williams inequality.

Proof of Theorem 12.8. Write $\theta_i = P_1(i), 0 \leq i \leq d$. By the property of P-polynomial scheme, there exists polynomials: $v_k(z)$ of degree k such that $P_k(i) = v_k(\theta_i)$.

By the definition, we have

$$u_1 = u_2 = \cdots = u_{t(\boldsymbol{u})} = 0.$$

We want to get a contradiction, assuming

$$s(\boldsymbol{u}Q) < \left[\frac{t(\boldsymbol{u})}{2}\right].$$

Let $s = s(\boldsymbol{u}Q)$ and $\{\nu_1, \nu_2, \ldots, \nu_s\} = \{i : \boldsymbol{u}Q_i \neq 0, i \neq 0\}$. Let $h(z)$ be a polynomial of degree $\left[\frac{t(\boldsymbol{u})}{2}\right] - s - 1 (\geq 0)$ such that $h(\theta_i) \neq 0$ for $i = 0, 1, \ldots, d$. Set

$$f(z) = h(z) \prod_{j=0}^{s} (z - \theta_{\nu_j})$$

with $\theta_{\nu_0} = \theta_0$. Then

$$f(\theta_i)\begin{cases} = 0 & \text{for } i \in \{0, \nu_1, \nu_2, \ldots, \nu_s\}; \\ \neq 0 & \text{for } i \notin \{0, \nu_1, \nu_2, \ldots, \nu_s\}. \end{cases}$$

Since $\deg(f(z)^2) = 2\left[\frac{t(\boldsymbol{u})}{2}\right]$,

$$f(z)^2 = b_0 v_0(z) + b_1 v_1(z) + \cdots + b_{2[\frac{t(\boldsymbol{u})}{2}]} v_{2[\frac{t(\boldsymbol{u})}{2}]}(z).$$

For each i with $2\left[\frac{t(\boldsymbol{u})}{2}\right] + 1 \leq i \leq d$, define $b_i = 0$ and set $\boldsymbol{b} = (b_0, b_1, \ldots, b_d)$. Since $2\left[\frac{t(\boldsymbol{u})}{2}\right] + 1 = t(\boldsymbol{u})$ or $t(\boldsymbol{u}) + 1$, we obtain

$$\boldsymbol{u} Q\, {}^t(\boldsymbol{b}\, {}^tP) = \boldsymbol{u} Q\, {}^tP \boldsymbol{b} = |X| \sum_{i=0} d u_i b_i = |X| u_0 b_0.$$

Since

$$(\boldsymbol{b}\, {}^tP)(\nu_j) = \sum_{k=0}^{d} b_k P_k(\nu_j) = \sum_{k=0}^{d} b_k v_k(\theta_{\nu_j}) = (f(\theta_{\nu_j}))^2 = 0, \quad 0 \leq j \leq s$$

So $\boldsymbol{u} Q\, {}^t(\boldsymbol{b}\, {}^tP) = 0$. Since $u_0 \neq 0$, $b_0 = 0$. Therefore

$$\sum_{i=0}^{d} f(\theta_i)^2 Q_i(0) = \sum_{i=0}^{d} \sum_{j=0}^{d} b_j P_j(i) Q_i(0) = |X| b_0 = 0.$$

Since $f(\theta_i)^2 > 0$, for $i \notin \{0, \nu_1, \nu_2, \cdots, \nu_s\}$ and $Q_i(0) = m_i > 0$. This is impossible. □

12.2 Codes in P-polynomial association schemes

Let $(X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a P-polynomial association scheme w.r.t. the ordering $A_0, A_1, \ldots, A_d$ and $k_1 = \theta_0 > \theta_1 > \cdots > \theta_d$ be the distinct eigenvalues of A_1. We define $\partial(x, y) = i$ if $(x, y) \in R_i$. Let δ be the minimum distance of Y, namely, $\delta = \min\{\partial(x, y) | x, y \in Y, x \neq y\} = t(\boldsymbol{a}_Y) + 1$. So $s^* = s(\boldsymbol{a}_Y^*) \geq \left[\frac{\delta - 1}{2}\right]$.

Definition 12.11: Perfect e-code

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a P-polynomial association scheme. A nonempty subset $Y \subset X$ is called a **perfect e-code** in $\mathfrak{X}$ if X can be partitioned as the following disjoint union

$$X = \bigcup_{y \in Y} \{x \in X | \partial(x, y) \leq e\}.$$

Since $|\{x \in X | \partial(x, y) \leq e\}| = \sum_{i=0}^{e} k_i$, if Y is a perfect e-code, then $|Y|(k_0 + k_1 + \cdots + k_e) = |X|$. If $\delta \geq 2e + 1$, then for two distinct points $y_1, y_2 \in Y$, we have $\{x \in X | \partial(x, y_1) \leq e\} \bigcap \{x \in X | \partial(x, y_2) \leq e\} = \emptyset$ which implies that $|Y|(k_0 + k_1 + \cdots + k_e) \leq |X|$.

Theorem 12.12: Lloyd theorem

Let $Y \subset X$ be a code in P-polynomial association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ and δ be the minimum diameter of Y. Set $e = \left[\frac{\delta - 1}{2}\right]$, then we have the following results.

(1) $|Y|(k_0 + k_1 + \cdots + k_e) \leq |X|$.

(2) If "=" holds in (1), then $s^* = s(\boldsymbol{a}_Y^*) = e$.

Moreover, the following polynomial

$$\Psi_e(z) = v_0(z) + v_1(z) + \cdots + v_e(z)$$

has e distinct zeros θ_k corresponding to $a_k^* \neq 0$, where $v_i(z)$ are the polynomials corresponding to the P-polynomial association scheme so that $A_i = v_i(A_1)$.

Proof. (1) We have proved that $M = \{0, \delta, \delta+1, \dots, d\}$. Consider the linear programming

$$0 \le \boldsymbol{a}_Y Q_k = \sum_{i=0}^{d} a_i Q_k(i) = \sum_{i \in M} a_i Q_k(i),$$
$$|Y| = \sum_{i \in M} a_i.$$

So $\boldsymbol{a}_Y$ is a program for (Q, M), since $a_0 = 1$ and $a_i \geqslant 0$ for $i \in M^*$.

Let $K_e := \Psi_e(\theta_0) = \sum_{j=0}^{e} P_j(\theta_0) = \sum_{j=0}^{e} k_j$ and define $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ by

$$\alpha_j = \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2.$$

Then $\alpha_0 = \left(\frac{\Psi_e(\theta_0)}{K_e}\right)^2 = 1$ and $\alpha_j \geqslant 0$ for $j \in D^*$. Note that

$$Q_j(i) = \frac{m_j P_i(j)}{k_i} = \frac{m_j v_i(\theta_j)}{k_i}.$$

So

$$\begin{aligned}(\alpha\, {}^tQ)(i) &= \sum_{j=0}^{d} \alpha_j Q_j(i) \\ &= \sum_{j=0}^{d} \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2 \frac{m_j v_i(\theta_j)}{k_i} \\ &= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} \Psi_e(\theta_j)^2 m_j v_i(\theta_j). \end{aligned} \tag{19}$$

The polynomial $\Psi_e(z)^2$ is of degree $2e$, then it has the following expansion.

$$\Psi_e(z)^2 = \sum_{\nu=0}^{2e} c_\nu v_\nu(z).$$

Since $e = \left[\frac{\delta-1}{2}\right]$, we have $2e = \delta - 1$ if δ is odd and $2e = \delta - 2$ if δ is even, namely, $2e < \delta$. Recall the second orthogonal relation in Theorem 3.3:

$$\begin{aligned}\delta_{\nu,i}|X|k_\nu &= \sum_{j=0}^{d} m_j P_i(j) P_\nu(j) \\ &= \sum_{j=0}^{d} m_j v_i(\theta_j) v_\nu(\theta_j).\end{aligned}$$

So for any $i \in M^*$, we have

$$\begin{aligned}(\alpha\, {}^tQ)(i) &= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} \left(\frac{\Psi_e(\theta_j)}{K_e}\right)^2 m_j v_i(\theta_j) \\ &= \frac{1}{K_e^2 k_i} \sum_{j=0}^{d} m_j \sum_{\ell=0}^{2e} c_\ell v_\ell(\theta_j) v_i(\theta_j) = 0.\end{aligned}$$

Therefore $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_d)$ is a program for Problem $(Q, M)'$. Next we want to show $c_0 = K_e$.

$$\sum_{\nu=0}^{d} m_\nu (\Psi_e(\theta_\nu))^2 = \sum_{\nu=0}^{d} m_\nu \sum_{i=0}^{e} \sum_{j=0}^{e} P_i(\nu) P_j(\nu) = \sum_{i=0}^{e} |X| k_i = |X| K_e.$$

On the other hand,

$$\sum_{\nu=0}^{d} m_\nu (\Psi_e(\theta_\nu))^2 = \sum_{\nu=0}^{d} Q_\nu(0) \sum_{i=0}^{2e} c_i P_i(\nu) = c_0 |X|.$$

Put $i = 0$ in Eq. (19).

$$\begin{aligned} \gamma = \sum_{j \in D} \alpha_j Q_j(0) &= \frac{1}{K_e^2 k_0} \sum_{j=0}^{d} \Psi_e(\theta_j)^2 m_j v_0(\theta_j) \\ &= \frac{1}{K_e^2} \sum_{j=0}^{d} \sum_{\nu=0}^{2e} c_\nu v_\nu(\theta_j) m_j v_0(\theta_j) \\ &= \frac{1}{K_e^2} \sum_{\nu=0}^{2e} c_\nu \sum_{j=0}^{d} m_j P_\nu(j) P_0(j) = \frac{1}{K_e^2} c_0 |X|. \end{aligned} \tag{20}$$

This implies that $\gamma = \frac{|X|}{K_e}$. Hence

$$|Y| = \sum_{i=0}^{d} a_i = \sum_{i \in M} a_i = g \leqslant \gamma = \frac{|X|}{K_e}.$$

This complete the proof of the first statement.

(2) If "=" holds in (1), then $\boldsymbol{a}$ is the max. program for (Q, M) and $\boldsymbol{\alpha}$ is the min. program for $(Q, M)'$. Using Eq. (17), we have

$$\alpha_j \left(\sum_{i=0}^{d} a_i Q_j(i)\right) = \alpha_j a_j^* = 0 \quad \text{for } j \in D^*.$$

So for each $j \in D^*$ with $a_j^* \neq 0$, we get $\alpha_j = 0$, i.e., $\Psi_e(\theta_j) = 0$. Note that $\Psi_e(z)$ is a polynomial of degree e and $\theta_0, \theta_1, \ldots, \theta_d$ are distinct real numbers. So $s^* = s(\boldsymbol{a}_Y^*) \leqslant e$. While by MacWilliam's inequality,

$$s^* = s(\boldsymbol{a}_Y Q) \geqslant \left\lceil \frac{t(\boldsymbol{a}_Y)}{2} \right\rceil = \left\lceil \frac{\delta - 1}{2} \right\rceil = e.$$

□

12.3 Designs in Q-polynomial association schemes

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a Q-polynomial association scheme and Y be a non-empty subset of X. Please recall that $\boldsymbol{a}_Y = (a_0, a_1, \ldots, a_d)$ and $\boldsymbol{a}_Y^* = \boldsymbol{a}_Y Q = (a_0^*, a_1^*, \ldots, a_d^*)$ are respectively the inner distribution and the dual distribution of Y. We shall first discuss the origin of concept of t-designs. Let $C \subseteq F_q^n$ be a linear code and its dual code is given by

$$C^\perp = \{y \in F_q^n : x \cdot y = 0, x \in C\}.$$

Define the weight enumerator of C as follows.

$$W_C(x, y) = \sum_{c \in C} x^{n - w(c)} y^{w(c)} = \sum_{i=0}^{n} a_i x^{n-i} y^i \in \mathbb{Z}[x, y],$$

where $w(c) = \#\{i : c_i \neq 0\}$ and $a_i = \#\{y \in C : w(y) = i\}$. The following MacWilliam's identity is well known and the proof is available from any book in coding theory.

$$W_{C^\perp}(x,y) = \frac{1}{|C|} W_C(x + (q-1)y, x - y).$$

Using MacWilliam's identity, we obtain the weight enumerator of $C^\perp$ as follows.

Theorem 12.13

Let Q be the second eigenmatrix of $H(n,q)$, then

$$W_{C^\perp}(x,y) = \sum_{i=0}^{n} a'_i x^{n-i} y^i,$$

where $\boldsymbol{a}' = \frac{1}{|C|}\boldsymbol{a}Q = (a'_0, a'_1, \ldots, a'_n)$ and $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$.

Proof.

$$\begin{aligned}
W_{C^\perp}(x,y) &= \sum_{i=0}^{n} a'_i x^{n-i} y^i \\
&= \frac{1}{|C|} W_C(x + (q-1)y, x - y) \\
&= \sum_{\nu=0}^{n} a_\nu (x + (q-1)y)^{n-\nu} (x-y)^\nu \\
&= \frac{1}{|C|} \sum_{\nu=0}^{n} a_\nu \sum_{\ell=0}^{n-\nu} \sum_{k=0}^{\nu} (-1)^{\nu-k} \binom{n-\nu}{\ell} \binom{\nu}{k} (q-1)^{n-\nu-\ell} x^{\ell+k} y^{n-\ell-k} \\
&= \frac{1}{|C|} \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} a_\nu \sum_{\ell=0}^{\mu} (-1)^{\nu-\mu+\ell} \binom{n-\nu}{\ell} \binom{\nu}{\mu-\ell} (q-1)^{n-\nu-\ell} x^{\mu} y^{n-\mu} \\
&= \frac{1}{|C|} \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} a_\nu \sum_{\ell=0}^{n-\mu} (-1)^{n-\mu-\nu-\ell} \binom{n-\nu}{\ell} \binom{\nu}{n-\mu-\ell} (q-1)^{n-\nu-\ell} x^{n-\mu} y^{\mu} \\
&= \frac{1}{|C|} \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} a_\nu \sum_{\ell=0}^{\mu} (-1)^{\ell} (q-1)^{\mu-\ell} \binom{n-\nu}{\mu-\ell} \binom{\nu}{\ell} x^{n-\mu} y^{\mu} \\
&= \frac{1}{|C|} \sum_{\mu=0}^{n} \sum_{\nu=0}^{n} a_\nu Q_\mu(\nu) x^{n-\mu} y^{\mu}.
\end{aligned}$$

□

Denote by $\boldsymbol{a}^*_C = \boldsymbol{a}_C Q = (a^*_0, a^*_1, \ldots, a^*_n)$ the dual distribution of C. The minimum distance of $C^\perp$ is defined by the maximum integer t so that $a^*_1 = a^*_2 = \cdots = a^*_t = 0$ and $a^*_{t+1} \neq 0$. The integer t here is called the strength and the above condition is equivalent that C is a t-design in $H(n,q)$. In 1973, Delsarte [Del73] generalized this concept to any Q-polynomial association scheme.

Definition 12.14: t-designs in a Q-polynomial association scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a Q-polynomial association scheme and Y be a non-empty subset of X. Let $\boldsymbol{a}^*_Y = (a^*_0, a^*_1, \ldots, a^*_d)$ be the dual distribution of Y. The subset Y is called a t-design in $\mathfrak{X}$ if $a^*_i = 0$ for $i = 1, 2, \ldots, t$.

Using Theorem 12.6, we have two equivalent conditions for t-designs.

(1) $E_i\psi_Y = 0$ for $i = 1, 2, \ldots, t$.

(2) $B_Y Q_i = 0$ for $i = 1, 2, \ldots, t$.

More generally, we may consider any subset $T \subset \{1, 2, \ldots, d\}$.

Definition 12.15: Designs of harmonic index T in a Q-polynomial association scheme

Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be a Q-polynomial association scheme. A non-empty subset $Y \subset X$ is called a ***T*-design** (or **design of harmonic index *T***) in $\mathfrak{X}$ if $a_i^* = 0$ for $i \in T$.

The integer t in Definition 12.14 is known as the strength of Y. Mac Williams inequality shows that the degree $s \geqslant \left[\frac{t}{2}\right]$.

Theorem 12.16

Let Y be a t-design in a Q-polynomial association scheme. Set $e = \left[\frac{t}{2}\right]$, then the following two statements hold.

(1) $|Y| \geqslant m_0 + m_1 + \cdots + m_e$.

(2) If "=" holds in (1), then $s = s(\boldsymbol{a}_Y) = e$. The polynomial

$$\Psi_e^*(z) = v_0^*(z) + v_1^*(z) + \cdots + v_e^*(z)$$

has e zeros θ_k^* corresponding to e indices k s.t. $a_k \neq 0$.

Note that $\theta_k^* = Q_i(k) = v_i^*(\theta_k^*)$,where $v_i^*(z)$ is a polynomial of degree i corresponding to the Q-structure.

We shall prove this theorem later using linear programming.

Definition 12.17

Let Y be a tight $2e$-design in a Q-polynomial association scheme. Then Y is called **tight** if $|Y| = m_0 + m_1 + \cdots + m_e$.

Proof of Theorem 12.16. (1) Suppose Y is a t-design and set $\boldsymbol{b} = \frac{1}{|Y|} = (b_0, b_1, \ldots, b_d)$. Then $b_0 = \frac{1}{|Y|}a_0^* = 1$ and $b_i = \frac{1}{|Y|}a_i^* \geqslant 0$ for $1 \leqslant i \leqslant d$. Let $M = \{0, t+1, \ldots, d\}$, then Y is a t-design if and only if $b_1 = b_2 = \cdots = b_t = 0$. For each $k \in D$,

$$\sum_{i\in M} b_i P_k(i) = \sum_{i=0}^{d} b_i P_k(i) = \sum_{i=0}^{d} \frac{1}{|Y|} \sum_{\ell=0}^{d} a_\ell Q_i(\ell) P_k(i) = \frac{|X|}{|Y|} a_k \geqslant 0. \tag{21}$$

So $\boldsymbol{b}$ is a program for the Problem (P, M) and $\boldsymbol{b}$ makes $g = \sum\limits_{i\in M} b_i$ maximal. Then $g = \frac{|X|}{|Y|}$ by putting $k = 0$ in Eq. (21). Suppose $K_e^* = m_0 + m_1 + \cdots + m_e$ and define

$$\beta_j = \left(\frac{\Psi_e^*(\theta_j^*)}{K_e^*}\right)^2, \quad \text{for } 0 \leqslant j \leqslant d.$$

Then $\beta_0 = \left(\frac{\Psi_e^*(\theta_0^*)}{K_e^*}\right)^2 = 1$, since $\Psi_e^*(\theta_0^*) = \sum\limits_{i=0}^{e} v_i^*(\theta_0) = \sum\limits_{i=0}^{e} m_i$. The polynomial of $(\Psi_e^*(z))^2$ is of degree, then

$$(\Psi_e^*(z))^2 = \sum_{\ell=0}^{2e} c_\ell^* v_\ell^*(z).$$

For each $i \in M^*$, we get

$$\begin{aligned}\sum_{j\in D}\beta_j P_j(i) &= \frac{1}{(K_e^*)^2 m_i}\sum_{j\in D}(\Psi_e^*(\theta_j^*))^2 Q_i(j)k_j \\ &= \frac{1}{(K_e^*)^2 m_i}\sum_{j\in D}\sum_{\ell=0}^{2e} c_\ell^* Q_\ell(j)Q_i(j)k_j \\ &= \frac{1}{(K_e^*)^2 m_i}\sum_{\ell=0}^{2e} c_\ell^* \sum_{j\in D} Q_\ell(j)P_j(i)m_i.\end{aligned}$$

Since $e = \left[\frac{t}{2}\right]$, if $i \in M^*$, then $i \geqslant t+1 > 2e$. This implies that $\sum_{j\in D}\beta_j P_j(i) = 0$. Therefore $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_d)$ is a program for the Problem $(P, M)'$ minimizing

$$\gamma = \sum_{j\in D}\beta_j P_j(0).$$

Next we will prove $c_0^* = K_e^*$. Calculate $\sum_{\ell=0}^{d} k_\ell(\Psi_e^*(\theta_\ell^*))^2$ in two different ways. On one hand, we have

$$\begin{aligned}\sum_{\ell=0}^{d} k_\ell(\Psi_e^*(\theta_\ell^*))^2 &= \sum_{\ell=0}^{d} k_\ell \sum_{i=0}^{e}\sum_{j=0}^{e} Q_i(\ell)Q_j(\ell) \\ &= \sum_{i=0}^{e}\sum_{j=0}^{e}\sum_{\ell=0}^{d} m_i P_\ell(i)Q_j(\ell) \\ &= |X|\sum_{i=0}^{e} m_i = |X|K_e^*.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{\ell=0}^{d} k_\ell(\Psi_e^*(\theta_\ell^*))^2 &= \sum_{\ell=0}^{d} k_\ell \sum_{j=0}^{2e} c_j^* Q_j(\ell) \\ &= \sum_{j=0}^{2e} c_j^* \sum_{\ell=0}^{d} P_\ell(0)Q_j(\ell) = |X|c_0^*.\end{aligned}$$

So $c_0^* = K_e^*$ and

$$\gamma = \sum_{j\in D}\beta_j P_j(0) = \frac{|X|c_0^*}{(K_e^*)^2} = \frac{|X|}{K_e^*}.$$

Therefore $\frac{|X|}{|Y|} = g \leqslant \gamma = \frac{|X|}{K_e^*}$, namely, $|Y| \geqslant m_0 + m_1 + \cdots + m_e$.

(2) If "=" holds in (1), then $\boldsymbol{b}$ is the maximum program for the Problem (P, M) and β is the minimum program for the Problem $(P, M)'$. By Lemma 11.6, we have

$$\beta_j\left(\sum_{i\in M} b_i P_j(i)\right) = 0, \quad \text{for } j \in D^*.$$

Since $\sum_{i\in M} b_i P_j(i) = \frac{|X|}{|Y|}a_j$, for each $j \in D^*$ with $a_j \neq 0$, we get $\beta_j = 0$, namely, $\Psi_e^*(\theta_j^*) = 0$. Since $\Psi_e^*(z)$ is a polynomial of degree e and $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are all distinct, we get $s \leqslant e$. Note that $s(\boldsymbol{a}_Y) = s(\boldsymbol{a}_Y QP) = s(a_Y^* P)$. By MacWilliam's inequality, $s = s(\boldsymbol{a}_Y) \geqslant \left[\frac{t(\boldsymbol{a}_Y^*)}{2}\right] = e$. This concludes the proof. ☐

12.4 Strength and degree of designs in Q-polynomial association schemes

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a Q-polynomial association scheme and Y be a non-empty subset of X. The strength t is the maximum positive integer such that $a_1^* = a_2^* = \cdots = a_t^* = 0$ and $a_{t+1}^* \neq 0$. The degree s of Y is defined by $s = |\{i : a_i \neq 0, 1 \leqslant i \leqslant d\}|$. We also call Y an ***s*-distance set**.

Theorem 12.18

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a Q-polynomial association scheme and Y be a non-empty subset of X with strength t and degree s. Set $e = \left[\frac{t}{2}\right]$. Then

(1) $|Y| \leqslant m_0 + m_1 + \cdots + m_s$.

(2) If "=" holds in (1), then $s = e$, namely, $|Y| = m_0 + m_1 + \cdots + m_e$.

Proof. (1) Since $\mathfrak{X}$ is symmetric, there exists an orthogonal matrix U which simultaneously diagonalizes all the matrices in the Bose-Mesner algebra $\mathfrak{A}$. Namely,

$$U^{-1} E_i U = \operatorname{diag}(0, \dots, 0, \underbrace{1, \dots, 1}_{m_i}, 0, \dots, 0) := T_i.$$

We may assume U is a matrix whose rows and columns are indexed respectively by X and X' with $|X| = |X'|$. Define $X_i' = \{x \in X : T_i(x, x) = 1\}$. Then $|X_i'| = m_i$ and $X' = X_0' \sqcup X_1' \sqcup \cdots \sqcup X_d'$. Let H_i be the matrix parametrized by $Y \times X_i'$ which is defined by

$$H_i(x, y) = \sqrt{|X|} U(x, y), \quad (x, y) \in Y \times X_i'.$$

Then H_0 is the all one column of length $|Y|$ and, for each $(x, y) \in Y \times Y$, we have

$$\begin{aligned} (H_i\, {}^t H_i)(x, y) &= \sum_{z \in X_i'} H_i(x, z) H_i(y, z) = |X| \sum_{z \in X_i'} U(x, z) U(y, z) \\ &= |X| \sum_{z \in X} U(x, z) T_i(x, x) U(y, z) = |X| E_i(x, y) \\ &= \sum_{\ell=0}^{d} Q_i(\ell) A_\ell(x, y) = \sum_{\ell=0}^{d} v_i^*(\theta_\ell^*) A_\ell(x, y), \end{aligned}$$

the last equality is obtained due to $\theta_\ell^* = Q_1(\ell)$ and $Q_i(\ell) = v_i^*(\theta_\ell^*)$. Since the degree of Y is s, for the inner distribution $\boldsymbol{a}_Y = (a_0, a_1, \dots, a_d)$, let $\{\ell_1, \ell_2, \dots, \ell_s\} \subset D^*$ such that $\{\ell_1, \ell_2, \dots, \ell_s\} = \{\ell : a_\ell \neq 0, 1 \leqslant \ell \leqslant d\}$. Take $M = \{0, \ell_1, \ell_2, \dots, \ell_s\}$ and consider the following polynomial.

$$F(z) = \frac{|Y|}{\prod_{i=0}^{s} (\theta_0^* - \theta_{\ell_i}^*)} \prod_{i=0}^{s} (z - \theta_{\ell_i}^*).$$

Then $F(\theta_0^*) = |Y|$ and $F(z)$ is a polynomial of degree s having zeros $\theta_{\ell_1}^*, \theta_{\ell_2}^*, \dots, \theta_{\ell_s}^*$. Set

$$F(z) = \sum_{j=0}^{s} f_j v_j^*(z), \quad f_j \in \mathbb{R}, \quad j = 0, 1, \dots, s.$$

Then we have

$$|Y| = F(\theta_0^*) = \sum_{j=0}^{s} f_j v_j^*(\theta_0^*) = f_0 m_0 + f_1 m_1 + \cdots + f_s m_s. \tag{22}$$

Consider the (x, y)-th entry of $\sum_{j=0}^{s} f_j H_j\, {}^t H_j$.

$$\begin{aligned} \sum_{j=0}^{s} f_j (H_j\, {}^t H_j)(x, y) &= \sum_{j=0}^{s} f_j \sum_{\ell=0}^{d} v_j^*(\theta_\ell^*) A_\ell(x, y) \\ &= \sum_{\ell=0}^{d} \sum_{j=0}^{s} f_j v_j^*(\theta_\ell^*) A_\ell(x, y) = \sum_{\ell=0}^{d} F(\theta_\ell^*) A_\ell(x, y). \end{aligned} \tag{23}$$

For $(x,y)\in Y\times Y$, the ℓ-the term in $\sum_{\ell=0}^{d}F(\theta^*_\ell)A_\ell(x,y)$ remains only these with $R_\ell\cap(Y\times Y)\neq\emptyset$. So, if we assume $\ell_0=0$, then

$$\begin{aligned}\sum_{\ell=0}^{d}F(\theta^*_\ell)A_\ell(x,y)&=\sum_{i=0}^{s}F(\theta^*_{\ell_i})A_{\ell_i}(x,y)\\&=F(\theta^*_0)A_0(x,y)=\delta_{x,y}|Y|.\end{aligned}\tag{24}$$

We construct a matrix H of size $|Y|\times(m_0+m_1+\cdots+m_e)$ by $H=[H_0,H_1,\ldots,H_s]$. Furthermore, let Λ be a matrix indexed by $\cup_{i=0}^{s}X'_i\times\cup_{i=0}^{s}X'_i$ such that $\Lambda(x,x)=f_i$ if $x\in X'_i$. It follows from Eqs. (23) and (24) that

$$H\Lambda\,{}^tH=|Y|I.$$

Then the rank of $H\Lambda\,{}^tH$ is $|Y|$. While the rank of $H\Lambda\,{}^tH$ is the sum of m_i such that $f_i\neq 0$ for $0\leqslant i\leqslant s$, so $|Y|\leqslant m_0+m_1+\cdots+m_s$.

(2) Suppose "=" holds in (1), that is $|Y|=m_0+m_1+\cdots+m_s$. Using Eq. (22),

$$m_0+m_1+\cdots+m_s=f_0m_0+f_1m_1+\cdots+f_sm_s.\tag{25}$$

Suppose H is a regular matrix, namely, $H\,{}^tH={}^tHH$. Then $\Lambda^{-1}=\frac{1}{|Y|}\,{}^tHH$. Moreover, ${}^tHH$ is positive definite which implies that $f_i>0$ for all $i=0,1,\ldots,s$. In particular, $f_0=1$. Next we want to show that $0<f_i\leqslant 1$ for $1\leqslant i\leqslant d$. For each $\ell\in M^*$, we have $F(\theta^*_\ell)=0$. Then $v^*_j(\theta^*_\ell)=Q_j(\ell)=0$, since $f_j>0$ for $0\leqslant j\leqslant s$. Here we define $f_j=0$ for $s+1\leqslant j\leqslant d$, so

$$\sum_{j\in D}f_jQ_j(\ell)=0.\tag{26}$$

We shall calculate

$$\sum_{\ell\in M}a_\ell Q_i(\ell)\sum_{j\in D}f_jQ_j(\ell)$$

in two different ways. Note that $a_0=1$ and $a_\ell=0$ for each $\ell\in D^*\backslash M$. By Eq. (22) and Eq. (26), we have

$$\begin{aligned}\sum_{\ell\in M}a_\ell Q_i(\ell)\sum_{j\in D}f_jQ_j(\ell)&=Q_i(0)\sum_{j\in D}f_jQ_j(0)+\sum_{\ell\in M^*}a_\ell Q_i(\ell)\sum_{j\in D}f_jQ_j(\ell)\\&=m_i\sum_{j\in D}f_jQ_j(0)=|Y|m_i.\end{aligned}\tag{27}$$

On the other hand, using Theorem 3.3 (7), Proposition 3.5 (3) and Proposition 12.2, we obtain

$$\begin{aligned}\sum_{\ell\in M}a_\ell Q_i(\ell)\sum_{j\in D}f_jQ_j(\ell)&=\sum_{\ell\in D}a_\ell\sum_{j\in D}f_j\sum_{k\in D}q^k_{ij}Q_k(\ell)\\&=\sum_{k\in D}\sum_{j\in D}f_jq^k_{ij}\sum_{\ell\in D}a_\ell Q_k(\ell)\\&=\sum_{j\in D}f_jq^0_{ij}\sum_{\ell\in D}a_\ell Q_0(\ell)+\sum_{k\in D^*}\sum_{j\in D}f_jq^k_{ij}\sum_{\ell\in D}a_\ell Q_k(\ell)\\&=f_im_i|Y|+\sum_{k\in D^*}\sum_{j\in D}f_jq^k_{ij}a^*_k.\end{aligned}\tag{28}$$

Eqs. (27) and (28) imply that

$$|Y|m_i(1-f_i)=\sum_{k\in D^*}\sum_{j\in D}f_jq^k_{ij}a^*_k.\tag{29}$$

Since every term in the RHS of Eq. (29) is non-negative, we get $0\leqslant f_i\leqslant 1$. It follows from Eq. (25) that $f_0=f_1=\cdots=f_s=1$. Moreover, Λ is the identity matrix which amounts to ${}^tHH=H\,{}^tH=|Y|I$, namely,

$${}^tH_iH_j=\begin{cases}|Y|I & \text{if } i=j,\\ \mathbf{0} & \text{if } i\neq j.\end{cases}\tag{30}$$

Let $\|A\|^2$ denote the sum of all entries squared of matrix A. Then

$$\begin{aligned}
\| {}^tH_iH_j\|^2 &= \sum_{x\in X'_i}\sum_{y\in X'_j}\left(({}^tH_iH_j)(x,y)\right)^2 = \sum_{x\in X'_i}\sum_{y\in X'_j}\left(\sum_{z\in Y}H_i(z,x)H_j(z,y)\right)^2\\
&= \sum_{x\in X'_i}\sum_{y\in X'_j}\left(\sum_{z\in Y}H_i(z,x)H_j(z,y)\right)\left(\sum_{w\in Y}H_i(w,x)H_j(w,y)\right)\\
&= \sum_{z,w\in Y}\left(\sum_{x\in X'_i}H_i(z,x)H_i(w,x)\right)\left(\sum_{y\in X'_j}H_j(z,y)H_j(w,y)\right)\\
&= \sum_{z,w\in Y}|X|^2E_i(z,w)E_j(z,w)\\
&= |X|\sum_{z,w\in Y}|X|(E_i\circ E_j)(z,w)\\
&= |X|\sum_{z,w\in Y}\sum_{\ell=0}^{d}q_{ij}^{\ell}E_\ell(z,w) = |X|\sum_{\ell=0}^{d}q_{ij}^{\ell}\,{}^t\psi_Y\,E_\ell\psi_Y. \qquad (31)
\end{aligned}$$

It follows from Eq. (30) that for any $0\leqslant i\neq j\leqslant s$ the identity $\sum\limits_{\ell=0}^{d}q_{ij}^{\ell}\,{}^t\psi_Y\,E_\ell\psi_Y=0$ holds. Since q_{ij}^{ℓ} are non-negative and ${}^t\psi_Y\,E_\ell\psi_Y=0\geqslant 0$, for $0\leqslant i\neq j\leqslant s$ and $\ell\in D$, we get ${}^t\psi_Y\,E_\ell\psi_Y=0$. In particular, $1\leqslant\ell\leqslant 2s-1$ if $i+j=\ell$. While $q_{i,j}^{i+j}>0$, then

$$ {}^t\psi_Y\,E_\ell\psi_Y=0,\quad \text{for } \ell=1,2,\ldots,2s-1.$$

Equivalently, $a_\ell^*=0$ for $\ell=1,2,\ldots,2s-1$. By putting $i=j=s$ in Eqs. (30) and (31), the following identity holds.

$$\begin{aligned}
m_s|Y|^2 &= \| {}^tH_sH_s\|^2\\
&= q_{s,s}^0|X|\left({}^t\psi_Y\,E_0\psi_Y\right)+|X|q_{s,s}^{2s}\left({}^t\psi_Y\,E_{2s}\psi_Y\right)\\
&= m_s|Y|^2+|X|q_{s,s}^{2s}\left({}^t\psi_Y\,E_{2s}\psi_Y\right).
\end{aligned}$$

However $q_{s,s}^{2s}>0$ since $\mathfrak{X}$ is a Q-polynomial association scheme, then ${}^t\psi_Y\,E_{2s}\psi_Y=0$. Finally, if $1\leqslant\ell\leqslant 2s$, then $E_\ell\psi_Y=\mathbf{0}$ which implies that $2s\leqslant t$. Wile $s\geqslant\left[\frac{t}{2}\right]$ by MacWilliam's inequality, so $s=\left[\frac{t}{2}\right]=e$. □

13 Classical t-designs and designs in Johnson association schemes

It is known that Johnson association schemes $J(v,k)$ are P- and Q-polynomial association schemes. In this section we will discuss the classical t-(v,k,λ) designs which are equivalent to t-designs in $J(v,k)$ for Q-structure.

Definition 13.1: Classical t-designs

Let v,k,λ be positive integers such that $1\leqslant k\leqslant v$. Let V be a finite set (as **points set**) of cardinality v and $\mathcal{B}\subset\binom{V}{k}$, the collection of some k-subsets of V which is called **block set**. The pair $(V,\mathcal{B})$ is called a $\boldsymbol{t}$**-**$\boldsymbol{(v,k,\lambda)}$ **design** if for each $T\subset\binom{V}{t}$ the following condition holds.

$$|\{B\in\mathcal{B}:T\subset B\}|=\lambda>0,$$

where λ is a constant which does not depend on the choice of T.

Proposition 13.2

If $(V, \mathcal{B})$ is a t-(v,k,λ) design, then, for $0 \leqslant s \leqslant t$, it must be an s-(v,k,λ_s) design with

$$\lambda_s = \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s} \lambda.$$

In particular, $|\mathcal{B}| = \lambda_0 = \binom{v}{t} \Big/ \binom{k}{t} \lambda$.

Proof. For each $S \in \binom{V}{s}$ with $0 \leqslant s \leqslant t$, define

$$\lambda(S) := \#\{B \in \mathcal{B} : T \subset B\}.$$

Calculating the number of paris of $(T, B) \in \binom{V}{t} \times \binom{V}{k}$ such that $S \subseteq T \subseteq B$ in two ways, we get

$$\lambda(S)\binom{k-s}{t-s} = \binom{v-s}{t-s}\lambda.$$

□

In the following, we use Y instead of $(V, \mathcal{B})$ to denote a non-empty subset of $\binom{V}{k}$. Next we will prove the equivalence between t-(v,k,λ) designs and t-designs in $J(v,k)$.

Theorem 13.3

Let $J(v,k) = (X, \{R_i\}_{0 \leqslant i \leqslant k}$ be Johnson association scheme and Y be a non-empty subset of X. Then Y is a t-(v,k,λ) design if and only if it is a t-design in $J(v,k)$ for Q-structure in the sense of Delsarte (given in Definition 12.14).

Proof. Suppose Y is a t-(v,k,λ) design. Construct a matrix indexed by $X \times \binom{V}{i}$ as follows:

$$M_i(x,\xi) = \begin{cases} 1 & \text{if } \xi \subset x; \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 13.2, for $0 \leqslant i \leqslant t$, we have

$$\lambda_i = \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}\lambda = |Y|\frac{\binom{v-i}{k-i}}{\binom{v}{k}} = \frac{|Y|}{|X|}\binom{v-i}{k-i}.$$

The characteristic vector of X and Y are respectively denoted by ψ_X and ψ_Y. Then for each $\xi \in \binom{V}{i}$, the following identity holds.

$$({}^tM_i\psi_Y)(\xi) = \sum_{x \in X} M_i(x,\xi)\psi_Y(x) = \sum_{y \in Y} M_i(y,\xi) = \#\{y \in Y : \xi \subset y\} = \lambda_i.$$

On the other hand,

$$({}^tM_i\psi_X)(\xi) = \sum_{x \in X} M_i(x,\xi) = \#\{x \in X : \xi \subset x\} = \binom{v-i}{k-i}.$$

Therefore ${}^tM_i\psi_Y = \frac{|Y|}{|X|}\,{}^tM_i\psi_X$. Please recall the following matrix C_i defined in Eq. (8).

$$C_i := \sum_{\ell=i}^{k} \binom{\ell}{i} A_{k-i}$$

It is proved in Proposition 8.6 that $C_i = M_i\,{}^tM_i$. So for each i with $0 \leqslant i \leqslant t$, it is true that $C_i\psi_Y = \frac{|Y|}{|X|}C_i\psi_X$. We have also mentioned in Proposition 8.4 that $\langle C_0, C_1, \ldots, C_r\rangle = \langle E_0, E_1, \ldots, E_r\rangle$ for each $r \in D = \{0, 1, \ldots, k\}$. Then we regard $J(v,k)$ as a O-polynomial association scheme w.r.t. the ordering of $E_0, E_1, \ldots, E_k$. So $E_i\psi_Y = \frac{|Y|}{|X|}E_i\psi_X$ for $i = 0, 1, \ldots, t$. In fact, if $1 \leqslant i \leqslant t$, then $E_i\psi_X = 0$ since ψ_X is the all one column vector and $E_iE_0 = 0$. So Y is a t-design in $J(v,k)$ for Q-structure in the sense of Delsarte.

Conversely, let Y be a t-design in $J(v,k)$ for Q-structure. We can assume that $E_i\psi_Y = \frac{|Y|}{|X|}E_i\psi_X$ for $0 \leqslant i \leqslant t$. In fact, if $i \neq 0$, then $E_i\psi_Y = 0$ and $E_0\psi_Y = \frac{|Y|}{|X|}\psi_X = \frac{|Y|}{|X|}\psi_X = \frac{|Y|}{|X|}E_0\psi_X$. From the above discussion, we have $M_i\,{}^tM_i\psi_Y = \frac{|Y|}{|X|}M_i\,{}^tM_i\psi_X$, i.e.,

$$(M_i\,{}^tM_i)\left(\psi_Y - \frac{|Y|}{|X|}M_i\,{}^tM_i\psi_X\right) = \mathbf{0}.$$

Equivalently, for $i = 0, 1, \ldots, t$, it is true that

$${}^tM_i\psi_Y = \frac{|Y|}{|X|}\,{}^tM_i\psi_X.$$

Note that $({}^tM_i\psi_Y)(\xi) = \#\{y \in Y : \xi \subset y\}$ and $({}^tM_i\psi_X)(\xi) = \#\{x \in X : \xi \subset x\} = \binom{v-i}{k-i}$, then

$$\lambda_i = ({}^tM_i\psi_Y)(\xi) = \frac{|Y|}{|X|}\binom{v-i}{k-i} = |Y|\frac{\binom{v-i}{k-i}}{\binom{v}{k}}$$

which means that Y is a i-(v,k,λ_i) design for $i = 0, 1, \ldots, t$. □

Similarly we can consider t-designs in Hamming association schemes $H(d,q)$. This concept is equivalent to so-called orthogonal array of strength t which was introduced by Rao [RR47] in 1947.

Definition 13.4: Orthogonal array

Let F be a finite set of cardinality q and set $X = F^d$. A non-empty subset $Y \subset X$ is called an **orthogonal array** of strength (at least) t if for each $z = (z_1, z_2, \ldots, z_d) \in X$ and a given set $L = \{\ell_1, \ell_2, \ldots, \ell_t\}$ the following identity holds.

$$|\{y = (y_1, y_2, \ldots, y_d) \in Y : y_\ell = z_\ell, \ell \in L\}| = \lambda > 0,$$

where λ is determined by t not depending on the choice of z and L.

14 Perfect e-codes and tight $2e$-designs

In this section, we will summarize the current results on the classification problem of perfect e-codes and tight $2e$-designs in $H(d,q)$ and $J(v,k)$.

14.1 Classification of perfect e-codes in $H(d,q)$

Perfect e-codes in $H(d,q)$ are very rare.

Example 14.1

(1) If $e \geqslant d$, then $|Y| = 1$ as trivial example.

(2) For any $e \in \mathbb{Z}^+$, then $|Y| = 2$ and Y is a d-cube for $d = 2e+1$ which is almost trivial.

(3) $e = 3$, $d = 23$, $q = 2$: binary Golay code with $|Y| = 2^{23}$.

(4) $e = 2$, $d = 11$, $q = 3$: ternary Golay code with $|Y| = 3^6$.

History of the study of perfect e-codes in $H(d,q)$.

(1) Tietäväinen-Perko [TP71] classified the perfect e-codes in $H(n,2)$ for each fixed e.

> **Theorem 14.2**
>
> Perfect e-codes are either trivial, Hamming codes of length $d = 2^{r-1}$ and error radius $e = 1$, or the single Golay code with $d = 23$ and $e = 3$.

(2) A long-standing conjecture was that there are no other linear or nonlinear perfect e-codes over any finite field. Van Lint [vL71] showed that this is true if $e \geqslant 3$ and $q = p^r$ with prime $p > e$ and $e \leqslant 7$.

(3) Tietäväinen [Tie73] proved there are no unknown perfect e-codes over finite fields for each fixed e. This argument mainly used Lloyd's polynomial and the condition $\sum_{i=0}^{e} \binom{n}{i} \mid 2^e$.

(4) Bannai [Ban77] proved that for each integer $e \geqslant 3$, there are at most finitely many perfect e-codes in $H(d,q)$ if $d > e$ and $q > 2$. He made use of Lloyd theorem but this proof cannot be applied for $e = 2$.

(5) Best [Bes83] classified perfect e-codes in $H(d,q)$ for $e \geqslant 3$ and arbitrary q except $e = 6$ and $e = 8$.

(6) Hong [Hon84] proved that there is no non-trivial perfect e-codes in the $H(d,q)$ for $e = 6$ or 8 and arbitrary q. Moreover, he [Hon86] proved the non-existence of unknown perfect e-code for $e \geqslant 3$ and $q \geqslant 3$ based on the so-called sphere packing condition and the integrality of the zeros of the Lloyd polynomial given by Bannai [Ban77]. It follows that tight $2e$-designs in the $H(n,q)$, i.e., orthogonal arrays of strength $2e$ that achieve the Rao bound, do not exist for $e \geqslant 3$ and $q \geqslant 3$.

(7) The classification problem is open only for $e = 1$ and $e = 2$.

   (a) Tietäväinen (1977) [Tie77] proved that there are no non-trivial perfect e-codes in $H(d,q)$ in the case that $e \geqslant 3$ and $q = p_1^{r_1} p_2^{r_2}$, where p_1 and p_2 are distinct primes, and r_1 and r_1 are positive integers.

   (b) Laakso [Laa79] extended this non-existence result to the case for $q = p_1^{r_1} p_2^{r_2} p_3^{r_3}$.

No non-trivial examples of perfect e-codes in $J(v,k)$ are known at all.

14.2 Classification of tight $2e$-designs in $J(v,k)$

The classification problem of tight $2e$-designs in $J(v,k)$ is as follows.

(1) If $e = 1$, there are many examples (i.e., symmetric 2-designs) and the classification seems to be hopeless.

(2) For $e = 2$, Ito [Ito75] proved that there are exactly two such designs, i.e., the 4-$(23,7,1)$ design and the 4-$(23,16,52)$ design.

(3) Peterson [Pet77] proved the non-existence of tight 6-designs.

(4) Xiang [Xiaed] proved the non-existence of tight 8-designs.

(5) For each fixed $e \geqslant 5$, Bannai [Ban77] showed that there are only finitely many tight $2e$-designs. Furthermore, Dukes–Short-Gershman [DSG13] showed the non-existence of tight $2e$-designs for $5 \leqslant e \leqslant 9$.

(6) It seems that the complete classification for $e \geqslant 10$ is still open.

Index

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